Mathematics Education: Creating and Seizing Opportunities

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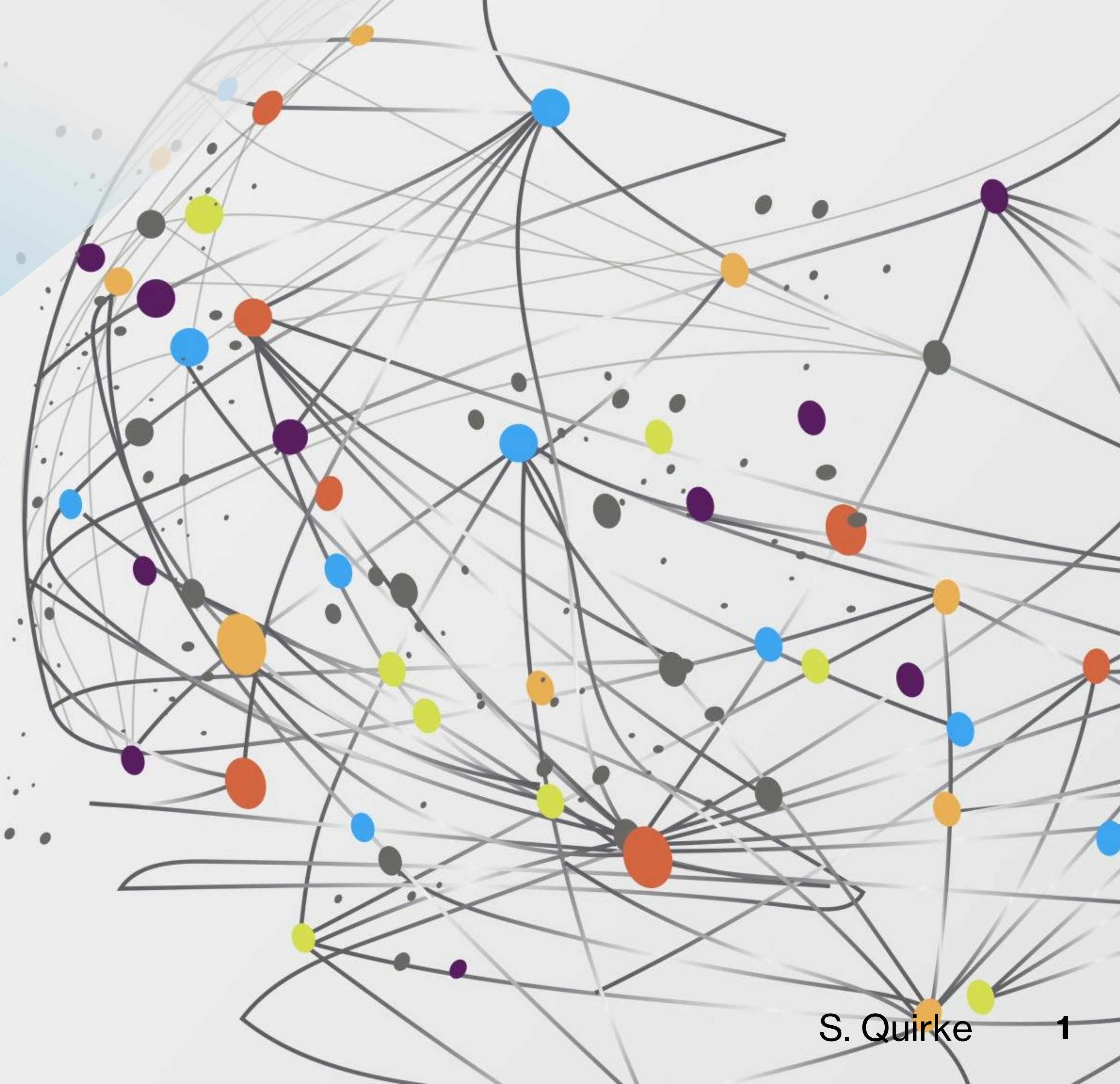
University of Galway



InterEducation Course Objectives

- The course aims to provide teachers with:
  - An insight into the teaching of mathematics in Irish schools
  - An opportunity to compare the teaching of mathematics in different European countries:
    - To exchange ideas and develop links with teachers from other European countries
  - New ideas and techniques for teaching mathematics and strategies for motivating learners.

what mathematics is...

what learning in mathematics is...

what helping someone to learn in mathematics is...

teaching mathematics is...

what mathematics is…	what learning in mathematics is…
what helping someone to learn in mathematics is…	teaching mathematics is…

→ Guides what we do…

what mathematics is...

The study of connected ideas and concepts that help to understand the world around us

what learning in mathematics is...

Coming to see why a topic/ concept/ idea is the way it is/ how to carry out a procedure:

- What is about
- How it is connected to other ideas/ concepts
- What is tells...

what helping someone to learn in mathematics is...

Providing someone with the opportunity/ environment for them to come to 'see' the mathematics idea/ concept/ procedure...

teaching mathematics is...

Providing and seizing opportunities that serve to enable the learner come to see (access) the topics/ ideas/ concepts/ procedures

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- How it is connected to other ideas/ concepts
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Being challenged to think and reason as to why/ how/ what…

what helping someone to learn in mathematics is…

Providing someone with the opportunity/ environment for them to **come to 'see'** the mathematics idea/ concept/ procedure…

teaching mathematics is…

Providing and seizing opportunities that serve to enable the learner **come to see** (access) the topics/ ideas/ concepts/ procedures

Teaching the division of fractions

Challenging learners to think…

The division of fractions…

- Discuss with the person beside you:
  - the approaches you adopt/ or would adopt when teaching the division of fractions...

The division of fractions…

- Discuss with the person beside you:
  - the approaches you adopt/ or would adopt when teaching the division of fractions...
  - The challenges, if any, that learner encounters when learning about the division of fractions..

The division of fractions…

- Discuss with the person beside you:
  - the approaches you adopt/ or would adopt when teaching the division of fractions...
  - The challenges, if any, that learner encounters when learning about the division of fractions..
  - How the ‘division of fractions’ is relevant to the mathematics that you teach...

Instructional Explanations

Research on prospective teachers' explanations of division with fractions, division by zero, and division with algebraic equations as procedural in nature, lacking regard for meaning and based on **memorisation** rather than understanding.

(Ball, 1988)

Research on prospective secondary school mathematics teachers in Ireland found that these pre-service teachers lacked **conceptual** understanding to support their teaching of the division of fractions.

(Slattery & Fitzmaurice, 2013)

↓

What potential challenges would this cause in the context of teaching

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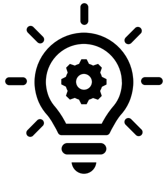

teaching mathematics is…

Providing and seizing opportunities that serve to enable the learner **come to see** (access) the topics/ ideas/ concepts/ procedures

How to create an environment guided by this...

what mathematics is...

The study of connected ideas and concepts that help to understand the world around us

what learning in mathematics is...

Coming to see why a topic/ concept/ idea is the way it is/ how to carry out a procedure:
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What is the role of memorisation in this environment for learning mathematics?

Instructional Explanations

Content-level understanding: This is *received knowledge* that is not ***actively*** *acquired* by learners.

Example:
Having been shown how to invert and multiply to find the answer, students would be unable to illustrate what to the division of fractions means

Concept-level understanding: The abstract ideas and clusters that define, bound, and ***guide inquiry*** in mathematics

Example:
Students operating at this level identify patterns and relationships.

To teach **vocabulary**

To give students a list of words and their definitions and ask students to demonstrate their understanding.

This is a fraction…

This is inverting a fraction…

.

.

.

Similar triangles … these are triangles which contain all the same angles of measure

The label ⟶ What it is (concept)

To teach **concepts**

To **create** a *problem* or *inquiry* situation where students can learn something about pattern finding/ be challenged to find a pattern that leads to the creation of a concept – **unearthing** a concept/ a mathematical idea/ procedure.

.

.

.

Discover in thinking mathematically

.

.

Creating the environment

Problem ⟶ pattern ⟶ concept ⟶ label

Instructional Explanations

Problem-solving level: The **analytic tools and methods** scholars and learners use to pose and resolve the puzzles, questions, and dilemmas of mathematics.

Strategies at this level:

- Inference
- Deductive thinking

Example:
Thinking abilities such as finding a pattern, working backwards, solving a similar problem, applying a procedure in situations to different from the one in which it was learned.

Instructional Explanations

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Concept-level understanding: The abstract ideas and clusters that define, bound, and ***guide inquiry*** in mathematics

Problem-solving understanding: The **analytic tools and methods** scholars and learners use to pose and resolve the puzzles, questions, and dilemmas of mathematics.

Instructional Explanations

Instrumental understanding: *rules without reasons*

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Problem-solving understanding: The **analytic tools and methods** scholars and learners use to pose and resolve the puzzles, questions, and dilemmas of mathematics.

Relational understanding: *knowing both what to do and why*

Tell me how and why…

$$\frac{1}{2} \div \frac{1}{3}$$

$$\frac{1}{2} \div \frac{1}{3}$$

Invert the second fraction and multiply

$$\frac{1}{2} \times \frac{3}{1}$$

What does invert mean?

Turn the second fraction upside down

Why is it the second fraction?

You’re dividing by the $\frac{1}{3}$

Is it always the second fraction?

Yes

Why do we multiply?

Because we inverted the fraction

How do we multiply?

$$\frac{1 \times 3}{2 \times 1}$$

Numerator by numerator
Denominator by denominator

$$\frac{3}{2}$$

Now practice a some more … textbook

$$\frac{1}{2} \div \frac{1}{3}$$

What is the **procedure** being studied here?

Invert the second fraction and multiply

$$\frac{1}{2} \times \frac{3}{1}$$

What does invert mean?

What **explanation** did the teacher give?

Turn the second fraction upside down

Why is it the second fraction?

You're dividing by the $\frac{1}{3}$

Is it always the second fraction?

What is the teacher **explaining**?

Yes

Why do we multiply?

Because we inverted the fraction

How do we multiply?

$$\frac{1 \times 3}{2 \times 1}$$

Numerator by numerator
Denominator by denominator

What is the **concept** being studied here?

$$\frac{3}{2}$$

Now practice a some more … textbook

Teaching for **instrumental understanding**

$$\frac{1}{2} \div \frac{1}{3}$$

Invert the second fraction and multiply

$$\frac{1}{2} \times \frac{3}{1}$$

What does invert mean?	Turn the second fraction upside down
Why is it the second fraction?	You're dividing by the $\frac{1}{3}$
Is it always the second fraction?	Yes
Why do we multiply?	Because we inverted the fraction
How do we multiply?	Numerator by numerator Denominator by denominator

$$\frac{1 \times 3}{2 \times 1}$$

$$\frac{3}{2}$$

Now practice a some more … textbook

Teaching for relational understanding:

Explanation, *Procedure* and *Concept*

Explanation	Why ‘something’ is the way it is...
Procedure	How we carry out ‘something’ in mathematics...
Concept	The mathematical idea...

The Division of Fractions

Explanation — Why 'something' is the way it is…

3 — The complete picture for dividing fractions – what the answer means.

Procedure — How we carry out 'something' in mathematics…

2 — Finding equivalent fractions | Multiplying fractions

Concept — The mathematical idea…

1

The key ideas of division:
- Division as repeated subtraction

The key ideas of fractions:
- A fraction as a ratio of two numbers

Planning the teaching of the division of fractions...

Generating tasks that focus on the key

3

1

The key ideas of division:
- Division as repeated subtraction

Task:

Repeatedly subtract 2 from 20.

Repeatedly subtract ½ from 20.

Repeatedly subtract ½ from $\frac{5}{2}$.

Repeatedly subtract ½ from $\frac{20}{8}$.

The key ideas of fractions:
- A fraction as a ratio of two numbers

Task:

Explore what does $6 \div 2$ mean..

Find two other numbers which have the same relationship

What about $\frac{1}{2}$ and the number 20? How do they compare?

Find two other numbers that have the same relationship

2

Finding equivalent fractions

Multiplying fractions

Generating tasks that focus on practicing the procedure

Task:

Find an equivalent fraction for $\frac{6}{2}$.

Find an equivalent fraction for $\frac{20}{\frac{1}{2}}$

Find an equivalent fraction for $\frac{20}{\frac{1}{2}}$, where the numerator is 1.

The complete picture

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Finding equivalent fractions

Multiplying fractions

Task:
Find an equivalent fraction for $\frac{6}{2}$.
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Concept-level understanding: The abstract ideas and clusters that define, bound, and ***guide inquiry*** in mathematics.

- Identifying patterns…

Problem-solving understanding: The **analytic tools and methods** scholars and learners use to pose and resolve the puzzles, questions, and dilemmas of mathematics.

- Thinking abilities

Create opportunities for learners to come to see…

Teaching for **relational understanding..**

$$\frac{1}{2} \div \frac{1}{3}$$

What is division?

What is a fraction?

What is a ratio?

What is proportion?

What is division?

Division as sharing

$$20 \div 5$$

Sharing €20 amongst 5 people.
How much does each person get?

Would this work for $\frac{1}{2} \div \frac{1}{3}$?

Division as repeated subtraction

$$20 \div 5$$

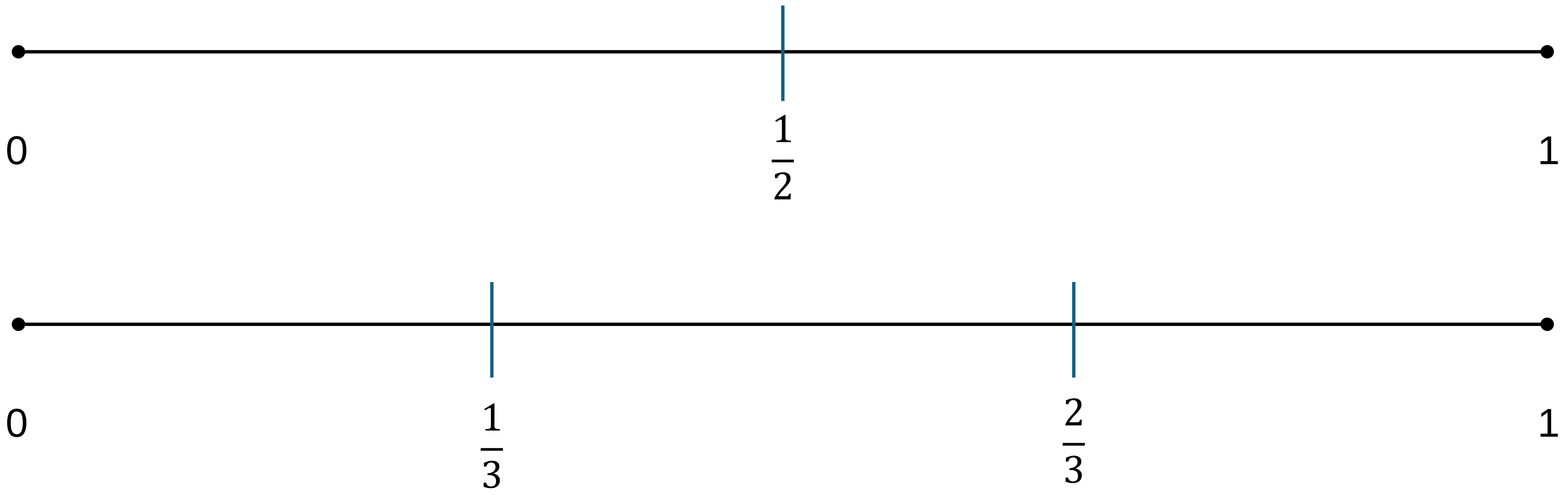
Repeatedly subtracting 5 from 20.
How many time can I do this?

Would this work for $\frac{1}{2} \div \frac{1}{3}$?

How could I show students $\frac{1}{2} \div \frac{1}{3}$?

Estimate the answer to $\frac{1}{2} \div \frac{1}{3}$

Will it be more than 1/ less than 1? A whole number?

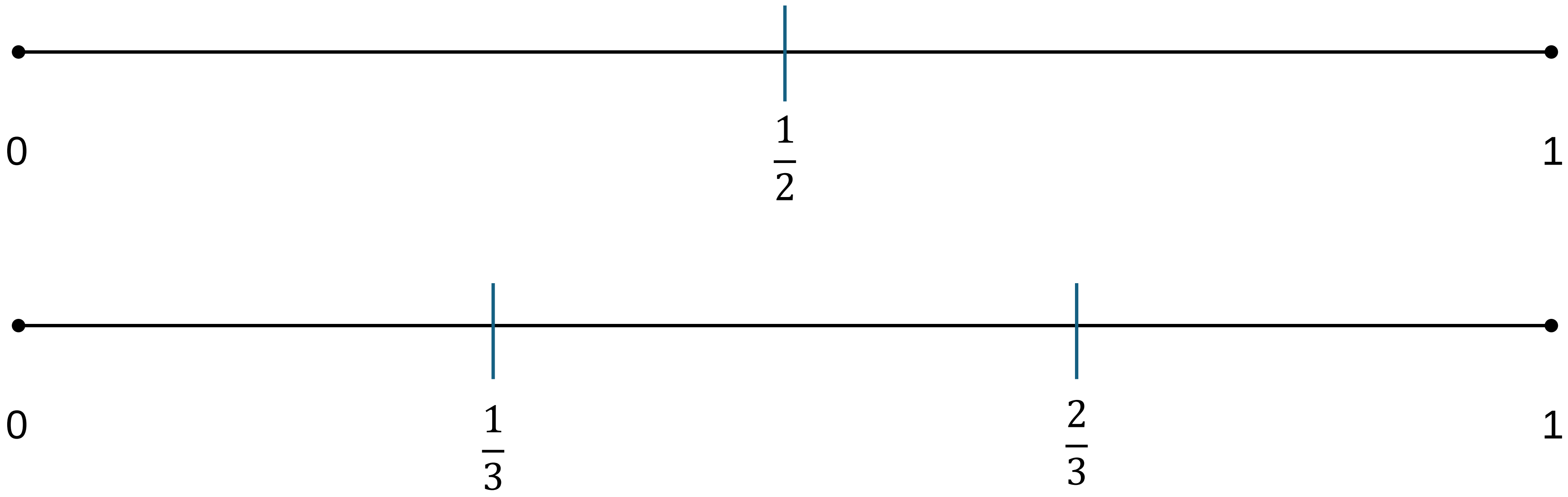


What are we working on here?

a) Procedure
b) Explanation
c) Concept

Estimate the answer to $\frac{1}{2} \div \frac{1}{3}$

Will it be more than 1/ less than 1? A whole number?



What does the answer $\frac{3}{2}$ mean?

Key Mathematical Idea

- Key mathematical ideas can form a framework for thinking about ‘important mathematics’.
- These ideas find application across all class/ year levels.
  - There may be difference in the complexity of their applications, but the ideas remain constant.

(Small, 2020)

Key Mathematical Ideas

$$\frac{1}{2} \div \frac{1}{3}$$

What is division?

What is a fraction?

What is a ratio?

What is proportion?

What is a fraction?

Part-Whole
Using the part-whole can be an effective starting point for building meaning of fractions.
Part-whole can be shading a region, part of a group of people, or part of a length.

Division
As with whole numbers, division means sharing into equal-sized groups.

Measurement
Measurement involves identifying a length and then using that length as a measurement unit to determine the length of an object.
The fraction $\frac{5}{8}$ is 5 times $\frac{1}{8}$.

Operator
Fractions can be used to indicate an operation, as in $\frac{4}{5}$ of 20 square metres. These situations indicate a fraction of a while number.

Ratio
The fraction $\frac{1}{4}$ can mean the probability of an event occurring is 1 in 4.
The ratio $\frac{3}{4}$ could be the ratio of those wearing jackets (part) to those not wearing jackets (part) [part : part ratio] or it could be those wearing jackets (part) to those in the class (whole) [part : whole ratio]

Key Mathematical Ideas

$$\frac{1}{2} \div \frac{1}{3}$$

What is division?

What is a fraction?

What is a ratio?

What is proportion?

A half is to a third… $\frac{1}{2} : \frac{1}{3}$

Fraction as division… $\dfrac{\frac{1}{2}}{\frac{1}{3}}$

What if I want the denominator to be 1… $\dfrac{\frac{1}{2}}{\frac{1}{3}}$

$$\dfrac{\frac{1}{2} \times \frac{3}{1}}{\frac{1}{3} \times \frac{3}{1}}$$

Equivalent fractions

Multiply the numerator and denominator by the same value.
Proportionality between (the ratio of) numerator and denominator remains the same.

$$\frac{1}{2} \div \frac{1}{3}$$

Invert the second fraction and multiply

$$\frac{1}{2} \times \frac{3}{1}$$

Consider what responses you would give to these questions

What does invert mean?

~~Turn the second fraction upside down~~

Why is it the second fraction?

~~You're dividing by the $\frac{1}{3}$~~

Is it always the second fraction?

~~Yes~~

Why do we multiply?

~~Because we inverted the fraction~~

How do we multiply?

~~Numerator by numerator~~
~~Denominator by denominator~~

$$\frac{1 \times 3}{2 \times 1}$$

$$\frac{3}{2}$$

~~Now practice a some more … textbook~~

Tell me how and why…

Teaching for **relational understanding**

$$\frac{1}{2} \div \frac{1}{3}$$

The procedure — Fraction division

The concept/ key ideas — Equivalent fractions

The concept/ key ideas — Knowing what a fraction is

The concept/ key ideas — Knowing what division is

Explanation: Why it is the way it is…

To prepare for…

Express as a single fraction:

$$\frac{5x-3}{2}-\frac{2x+1}{3}$$

Equivalent fractions

Junior cycle higher level

To prepare for

Solve the equation:

$$\frac{3x-1}{6}-\frac{x-3}{4}=\frac{4}{3}$$

Equivalent fractions

Junior cycle higher level

To prepare for…

Simplify:

$$\frac{x^{\frac{1}{2}} + x^{-\frac{1}{2}}}{x^{\frac{1}{2}}}$$

Equivalent fractions

Division

Senior cycle higher level

To prepare for…

Simplify:

$$\frac{x-5}{x+1} \div \frac{x^2-25}{x^2+4x+3}$$

Equivalent fractions
Division

Senior cycle higher level

To prepare for…

Simplify:

$$\frac{1-\frac{9}{x^2}}{2+\frac{6}{x}}$$

Equivalent fractions

Division

Senior cycle higher level

Mathematical proficiency

Explanation — Why ‘something’ is the way it is…

Procedure — How we carry out ‘something’ in mathematics…

Procedural Knowledge

- Allow mathematical tasks to be completed efficiently.

Concept — The mathematical idea…

Conceptual Knowledge

- Procedures depend upon principles represented conceptually.

Both kinds of knowledge are crucial.
Focus on how they are related, as opposed to which is more important.
(Hiebert and Carpenter, 1992)

The role of the mathematics teacher

Teacher as intermediary

What does a mathematics teacher do:

- Discuss with the person beside you all of the aspects of teaching mathematics:
  - What does teaching mathematics involve?
  - Job description…

Mathematics teaching analysis

- Core tasks:
  - Setting and clarifying goals
  - Evaluating a textbook's approach to a topic
  - Selecting and designing a task
  - Re-scaling tests
  - Choosing and using representations
  - Analysing and evaluating student responses
  - Analysing and responding to student errors
  - Managing productive discussions
  - Figuring out what students are learning

(Bass & Ball, 2004)

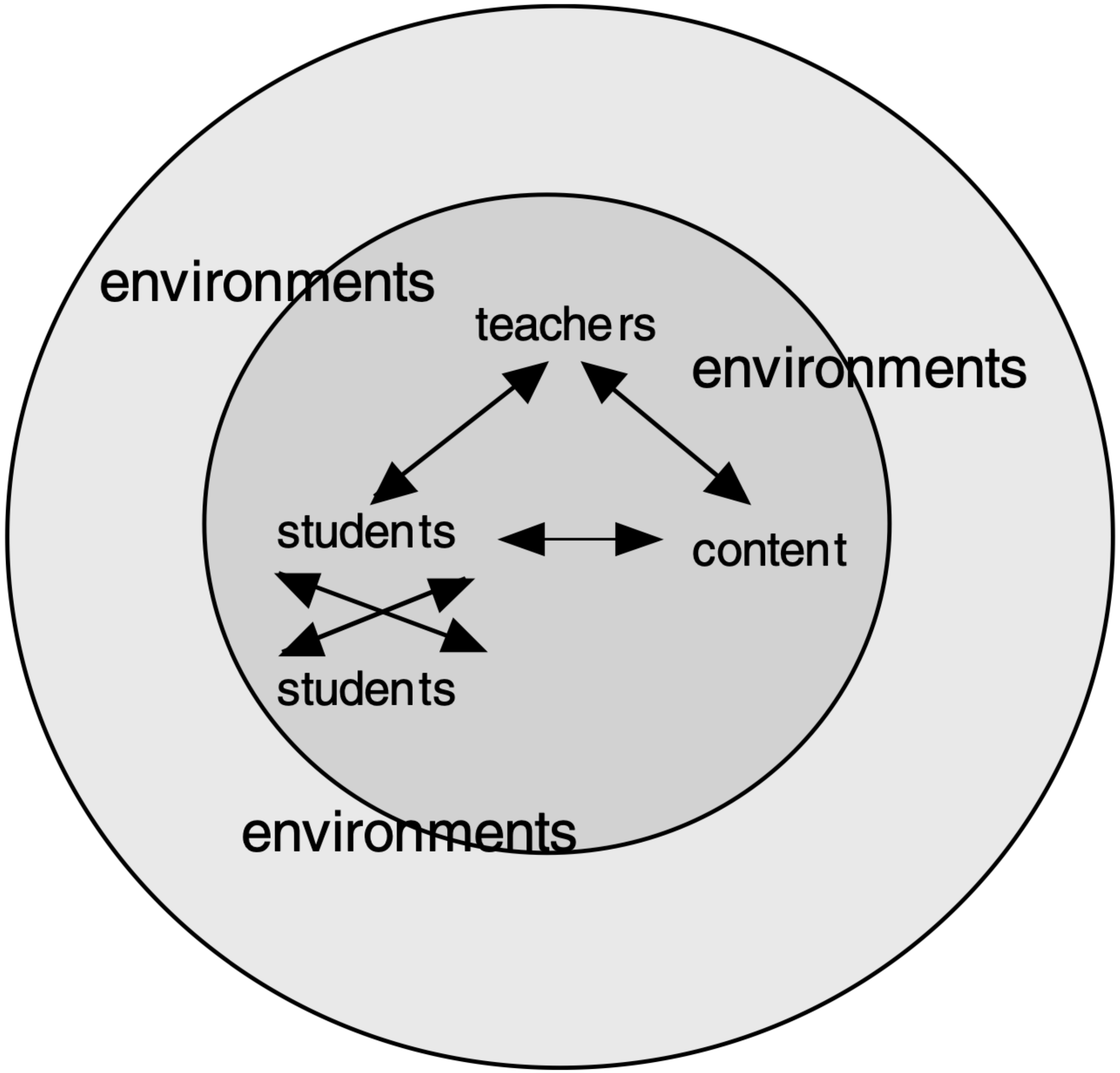
Environments: other teachers and students, school leaders, parents, assessments...

Instruction as interaction

- Teaching is what teachers do, say and think with learners, concerning *content*, in particular organisations and other **environments**, in time.
- Teaching is a collection of practices, including pedagogy, learning, instructional design and managing organisation.

(Cohen et al., 2003)

Instruction as interaction



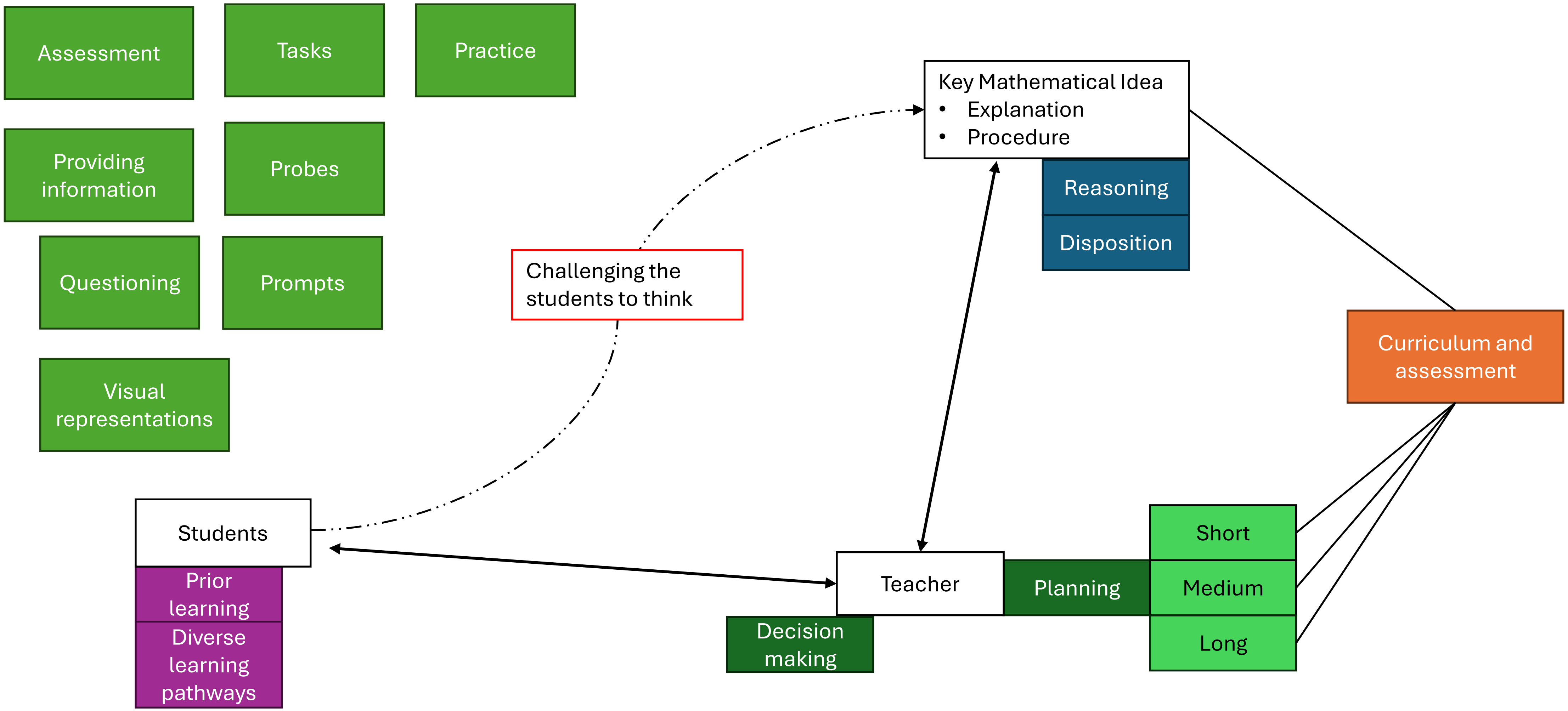
(Cohen et al., 2003)

Acting as an **intermediary**

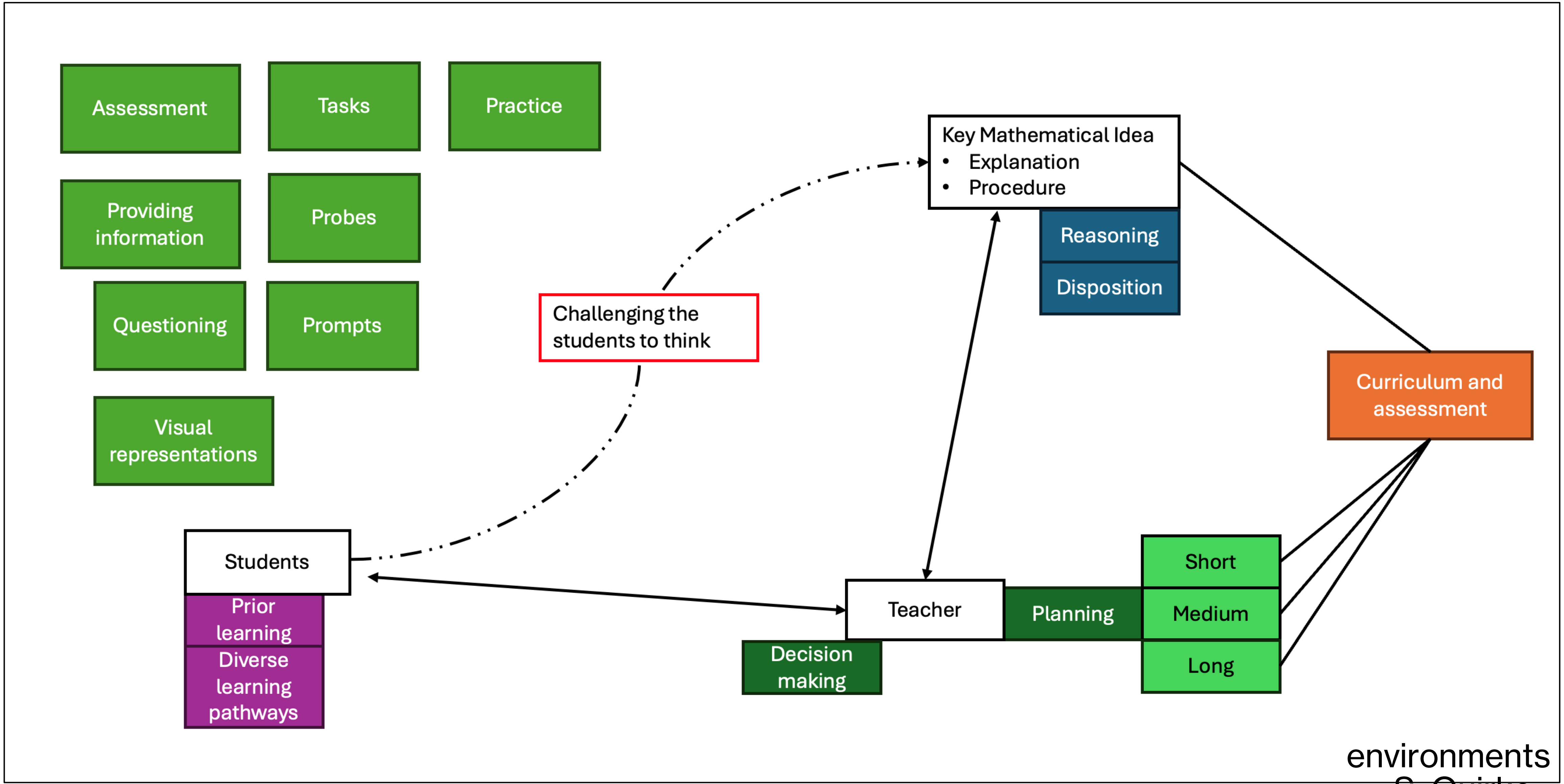
... situated, acting, or coming between.

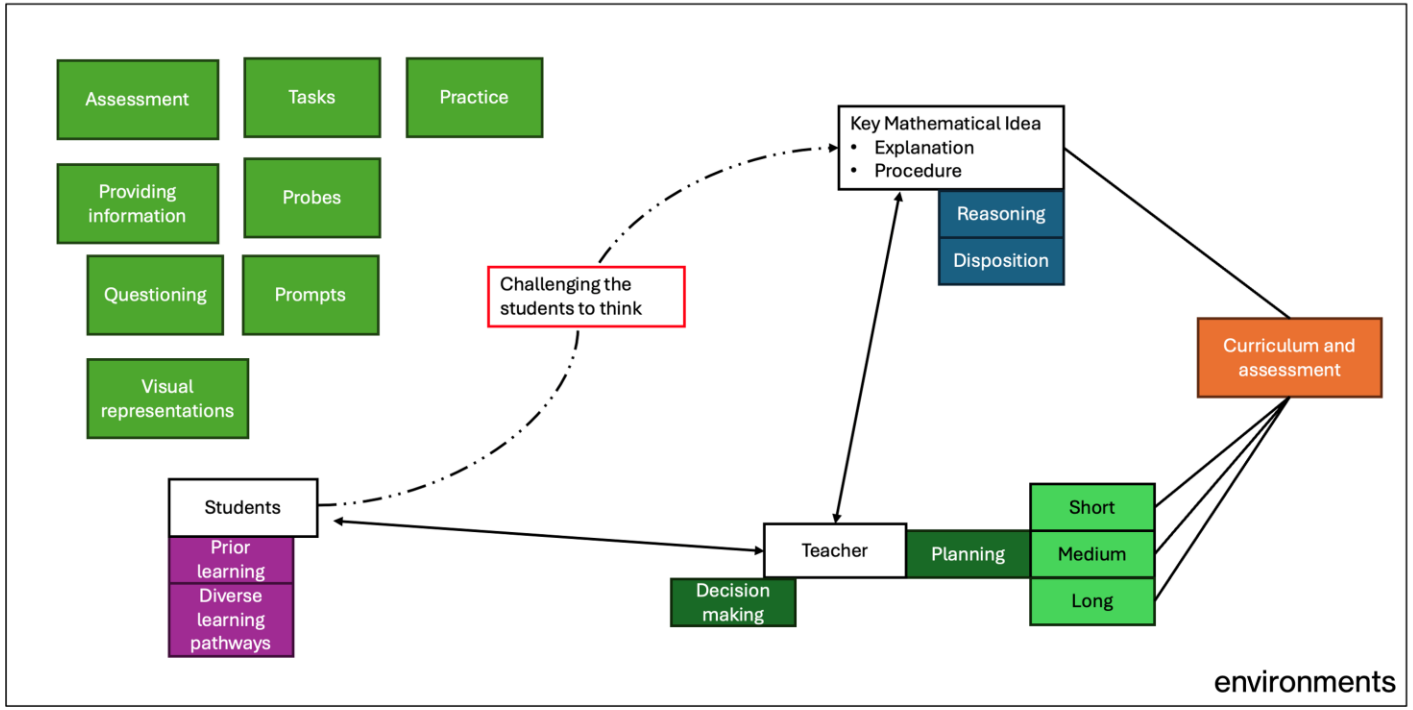
... something that acts a medium or means.

Collins Dictionary


Assessment
Tasks
Practice
Providing information
Probes
Questioning
Prompts
Visual representations
Challenging the students to think
Key Mathematical Idea
• Explanation
• Procedure
Reasoning
Disposition
Curriculum and assessment
Students
Prior learning
Diverse learning pathways
Teacher
Planning
Decision making
Short
Medium
Long

Teacher *as* intermediary





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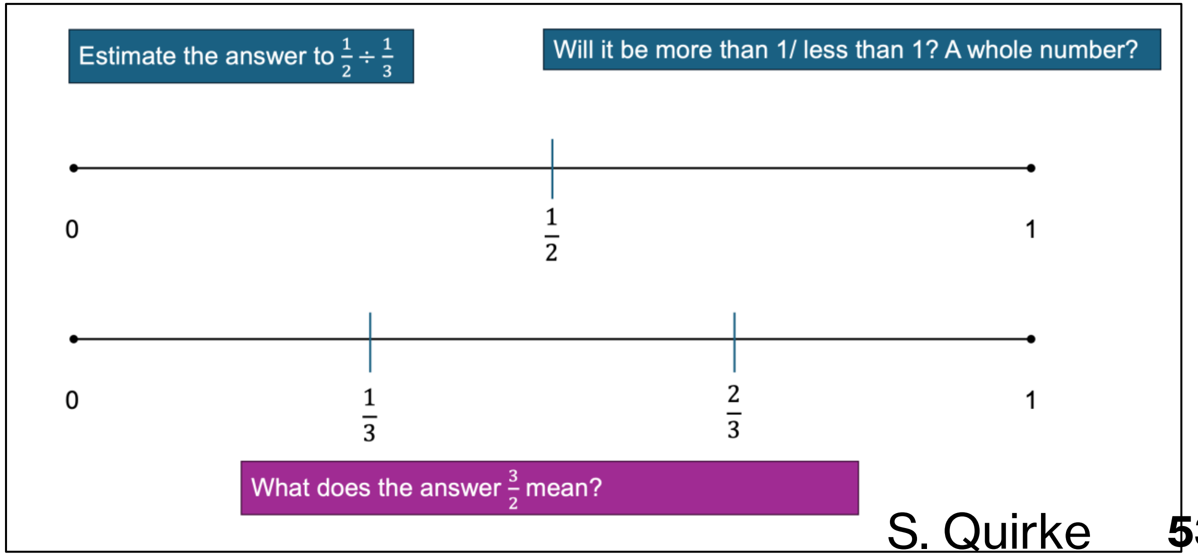


Task

Order the following fractions from smallest to largest:

$$\frac{99}{100}, \frac{6}{7}, \frac{15}{16}$$

Task 1

Order the following fractions from smallest to largest:

$$\frac{99}{100}, \frac{6}{7}, \frac{15}{16}$$

Task 2

Develop a convincing argument to support your order.

You may consider using visual representations to support your argument.

Task

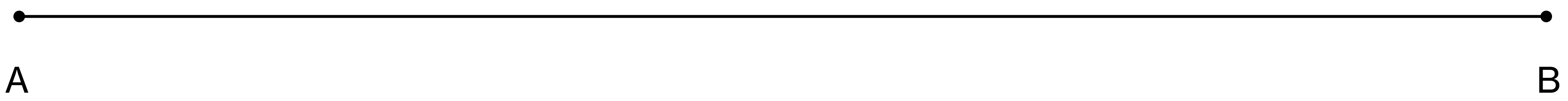
On the number line, place a line where you think $\frac{1}{100}$ is.

Task

On the number line, place a line where you think $\frac{1}{7}$ is.

Task

On the number line, place a line where you think $\frac{1}{16}$ is.



Task

Three people are completing a journey from A to B.
It takes John 100 steps to complete the journey A to B.
It takes Emma 7 steps to complete the journey from A to B.
It takes Sinéad 16 steps to complete the journey from A to B.

a. Who has the largest/ shortest stride?
b. If John has 99 steps completed, Emma has 6 steps completed, and Sinéad has 15 steps completed, who is closest to Point B?

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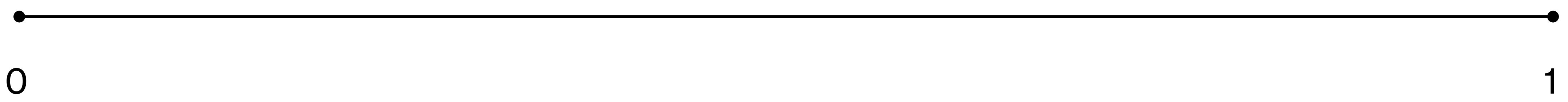


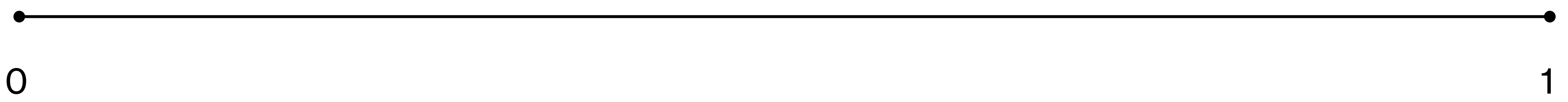
Task

On the number line, place a line where you think $\frac{1}{100}$ is.

- What do you have to do the line to find $\frac{1}{100}$.

Divide the line into 100 pieces/ steps/ jumps.

To find $\frac{1}{100}$ we must do $1 \div 100$



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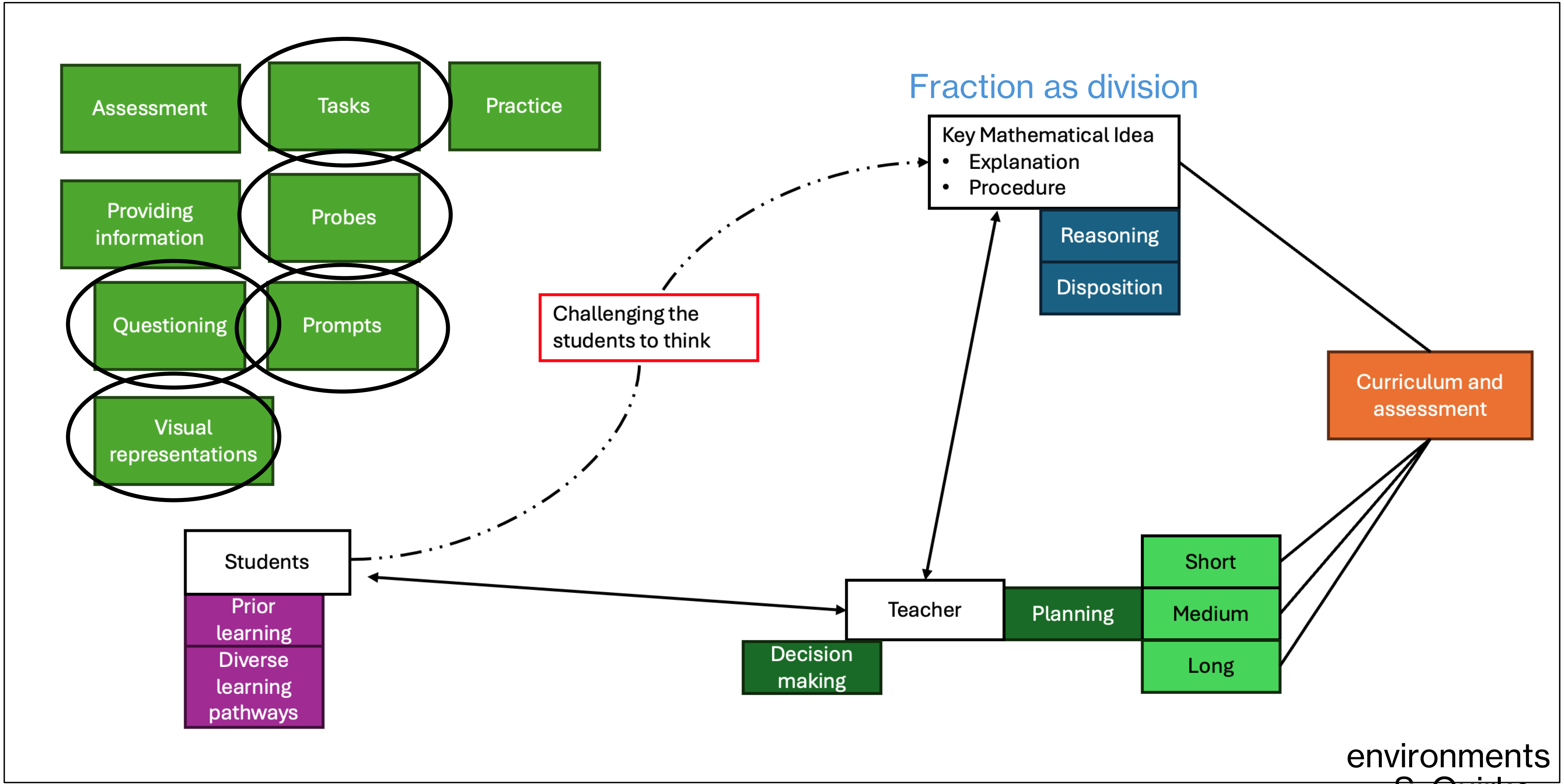
Fraction as division

Teacher *as* intermediary



environments
S. Quirke

Seizing Opportunities

Teaching for the now, and the future

Seize (an opportunity): You **take advantage** of a situation

What approaches are used to teach:

$$13 \times 27$$

How can we build on this in secondary school?

13×27

$$
\begin{array}{r}
{}^{2}13 \\
\times \quad 27 \\
\hline
91 \\
{}^{1}260 \\
\hline
351
\end{array}
$$

$$13 \times 27$$

$$\begin{array}{r} {}_{2}13 \\ \times\ 27 \\ \hline 91 \\ {}^{1}260 \\ \hline 351 \end{array} \rightarrow \begin{array}{r} 13 \\ \times 27 \\ \hline \end{array}$$

$$7(3) + 7(10) = 21 + 70 = 91$$

$$20(3) + 20(10) = 60 + 200 = 260$$

$$\overline{351}$$

$$13 \times 27$$

$$\begin{array}{r} 13 \\ \times\ 27 \\ \hline 91 \\ 260 \\ \hline 351 \end{array}$$

→

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$$7(3) + 7(10) = 21 + 70 = 91$$

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$$351$$

↓

$$7(3+10) + 20(3+10)$$

Distributive property

$$13 \times 27$$

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Distributive property

$$(7+20)(3+10) = (27)(13)$$

$$13 \times 27$$

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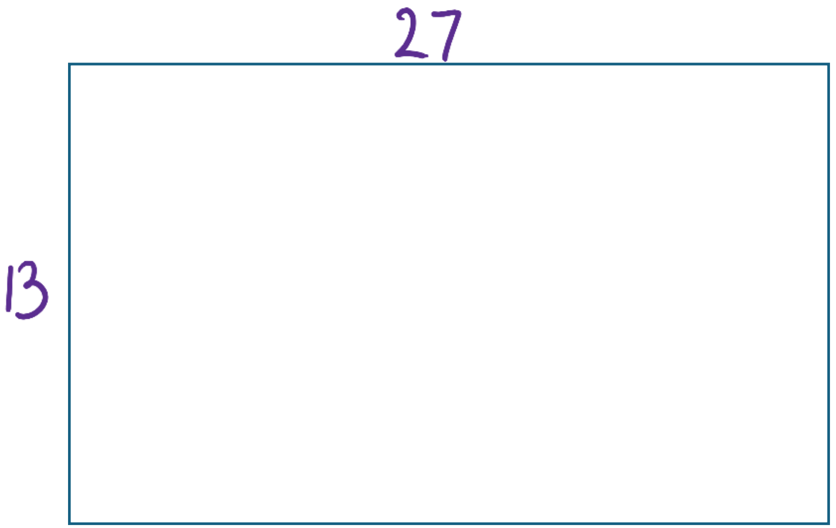
↓

$7(3+10) + 20(3+10)$ **Distributive property**

↓

$(7+20)(3+10) = (27)(13)$



$$13 \times 27$$

$$\begin{array}{r} 13 \\ \times\ 27 \\ \hline 91 \\ 260 \\ \hline 351 \end{array}$$

$\rightarrow$

$$\begin{array}{r} 13 \\ \times\ 27 \\ \hline \end{array}$$

$7(3) + 7(10) = 21 + 70 = 91$

$20(3) + 20(10) = 60 + 200 = 260$

351

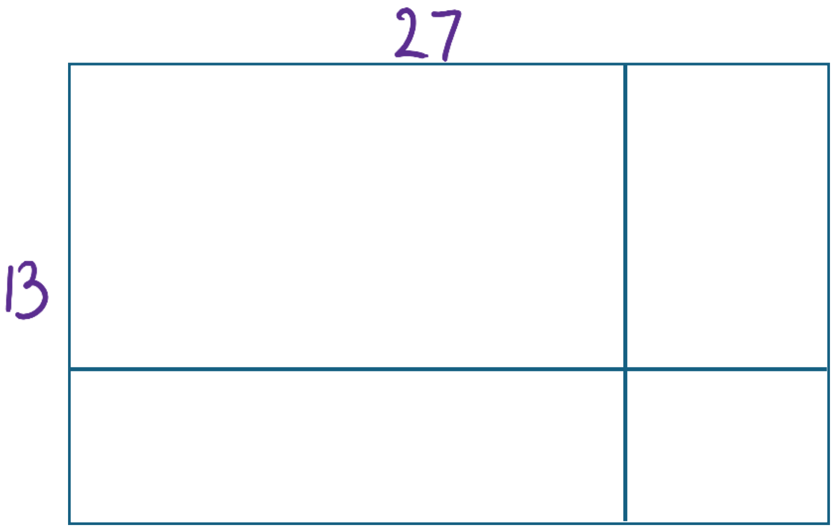
$\downarrow$

$7(3+10) + 20(3+10)$

Distributive property

$\downarrow$

$(7+20)(3+10) = (27)(13)$



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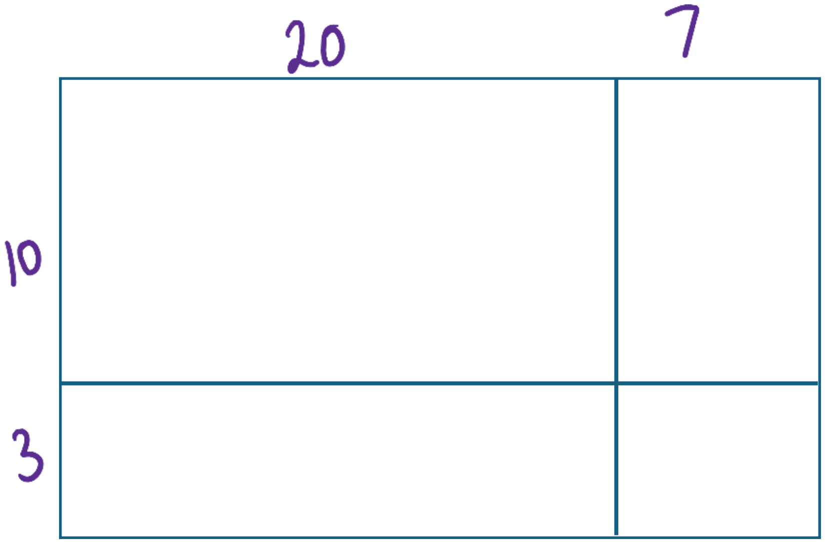

$$13 \times 27$$

$$\begin{array}{r} {}^{2}13 \\ \times \;\; 27 \\ \hline 91 \\ {}^{1}260 \\ \hline 351 \end{array}$$

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$7(3) + 7(10) = 21 + 70 = 91$

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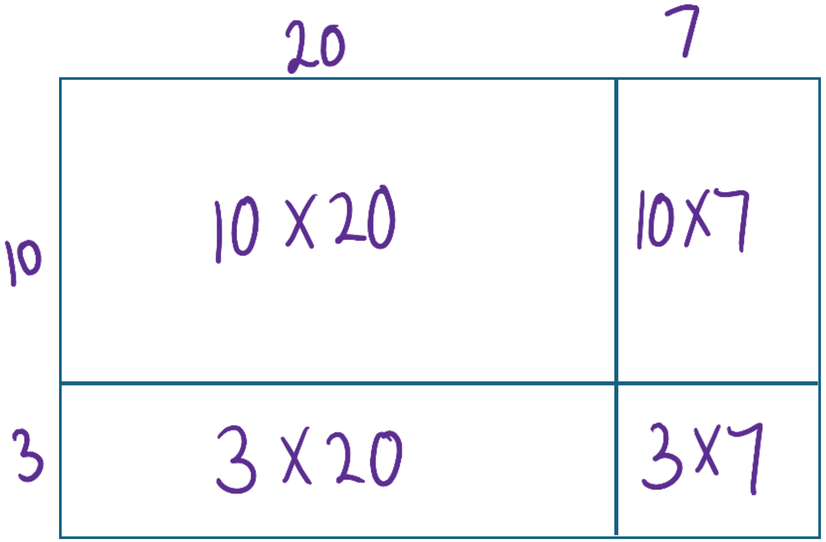
351

$$7(3+10) + 20(3+10)$$

Distributive property

$$(7+20)(3+10) = (27)(13)$$

Array diagram



To prepare for…

Solve the equation:

$$3(x + 7)$$

Distributive property

Junior cycle ordinary level

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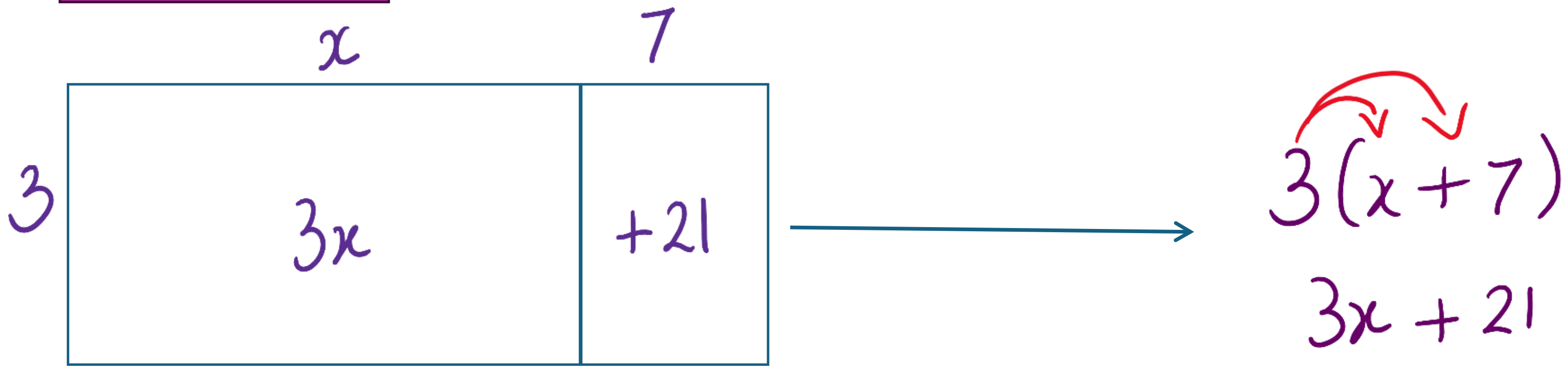
Solve the equation:

$$3(x+7)$$

Distributive property

Junior cycle ordinary level

Array diagram



To prepare for…

Solve the equation:

$$(x-3)(x-7)$$

Distributive property

The product of two negative numbers

Junior cycle Higher Level

To prepare for…

Solve the equation:

$$(x-3)(x-7)$$

Distributive property
The product of two negative numbers

Junior cycle Higher Level

Array diagram

	x	-3
x	x^2	$-3x$
-7	$-7x$	21

$$x(x-7)-3(x-7)$$

$$x^2-7x-3x+21$$

$$x^2-10x+21$$

To prepare for…

Solve the equation:

$$-y(2y - 3x) - y(xy + 3x)$$

Distributive property

The product of two negative numbers

Senior Cycle Higher Level

Seize (an opportunity): You **take advantage** of a situation

Connections to future learning

Mathematical learning context

Enabling learners to ‘come to see” why and what

Creating and seizing opportunities

Guiding towards how *and why...*

The slope of a line

Explanation

Why 'something' is the way it is...

3

The complete picture for finding the slope of a line – what the answer means.

Procedure

How we carry out 'something' in mathematics...

2

Task: Complete this for finding the slope of a line. Consider:

- Key ideas
- The procedure

Concept

The mathematical idea...

1

The slope of a line

Explanation

Why ‘something’ is the way it is...

3

The complete picture for finding the slope of a line – what the answer means.

Procedure

How we carry out ‘something’ in mathematics...

2

Labelling points

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Concept

The mathematical idea...

1

The gradient of a line.
- The relationship between the vertical change and horizontal change.

The difference between two points is found using subtraction.

You display these images to the students. What key ideas are you trying to probe for:

You display these images to the students. What key ideas are you trying to probe for:

In which photo will the cyclists most likely be traveling at the fastest speed?

Why do you think they will be travelling at the fastest speed in this photo?

In which photo will they be exerting the most energy?

Why do you think they will be exerting the most energy in photo?

Steepness (rising or falling in altitude) over a certain distance

The mathematical idea…

Concept

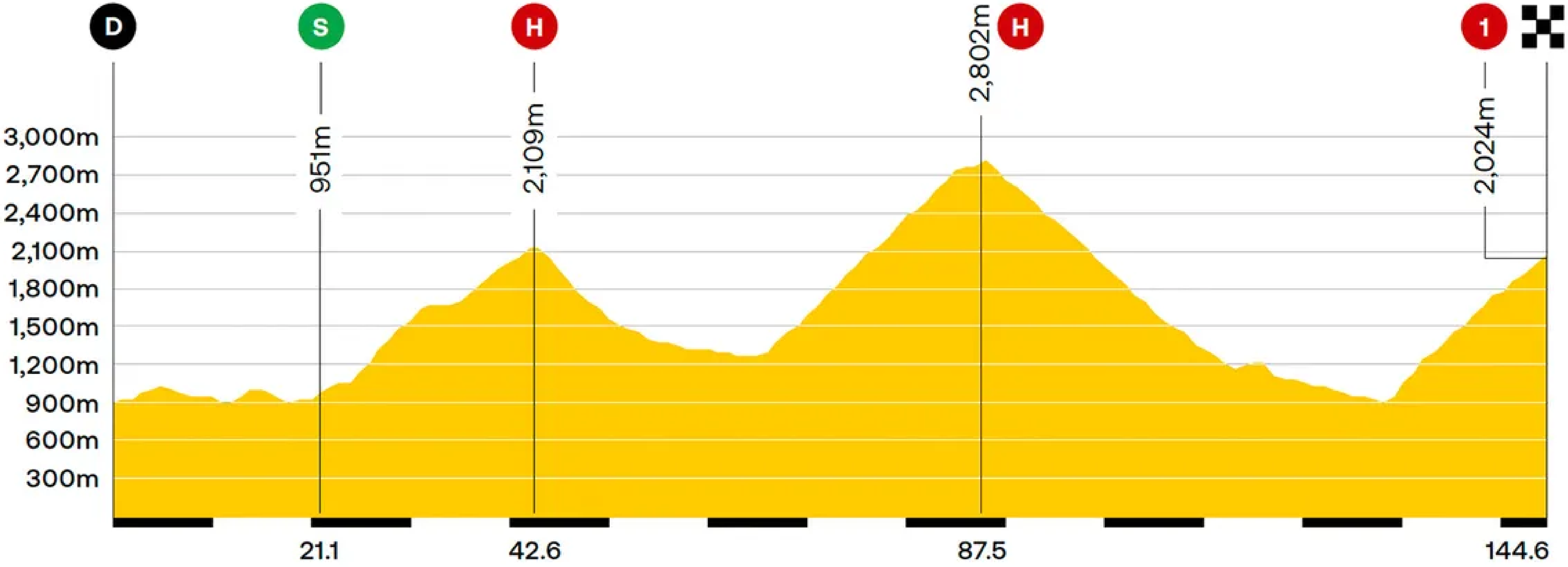
1

The gradient of a line.
- The relationship between the vertical change and horizontal change.

The difference between two points is found using subtraction.

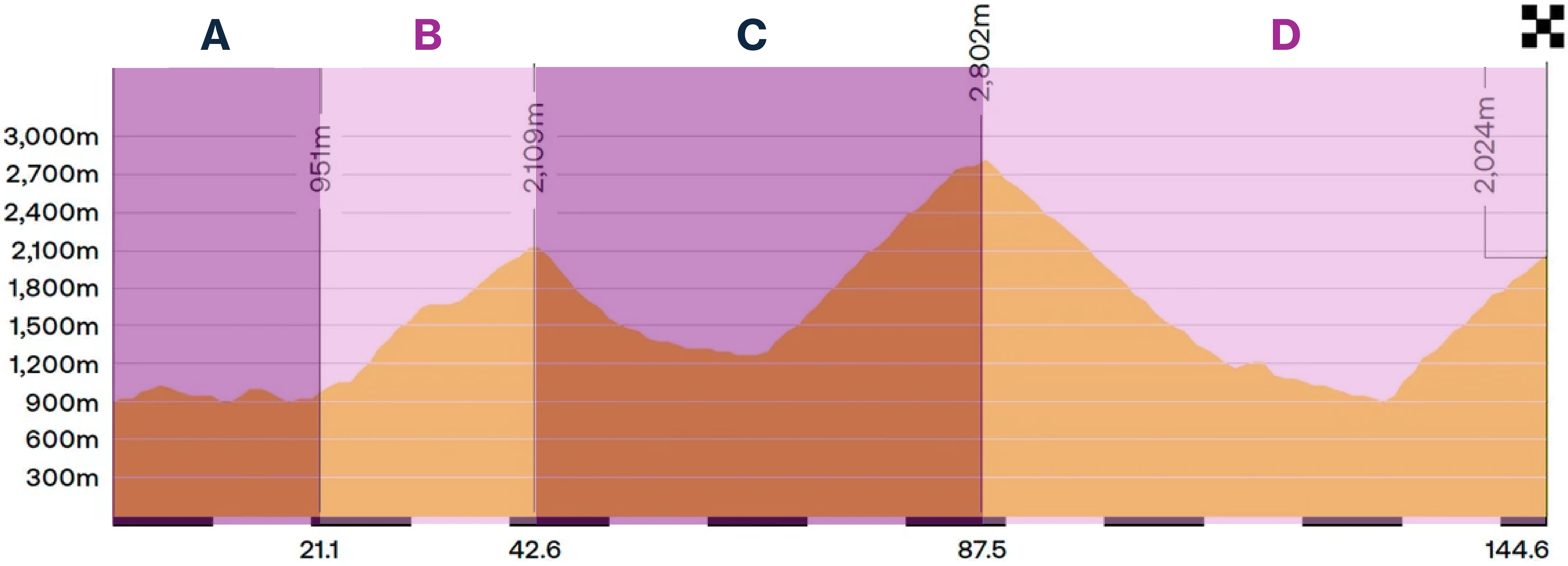
S. Quirke

This is stage 19 of the Tour de France, Embrun to Isola



Consider questions that you could pose to students?

This is stage 19 of the Tour de France, Embrun to Isola



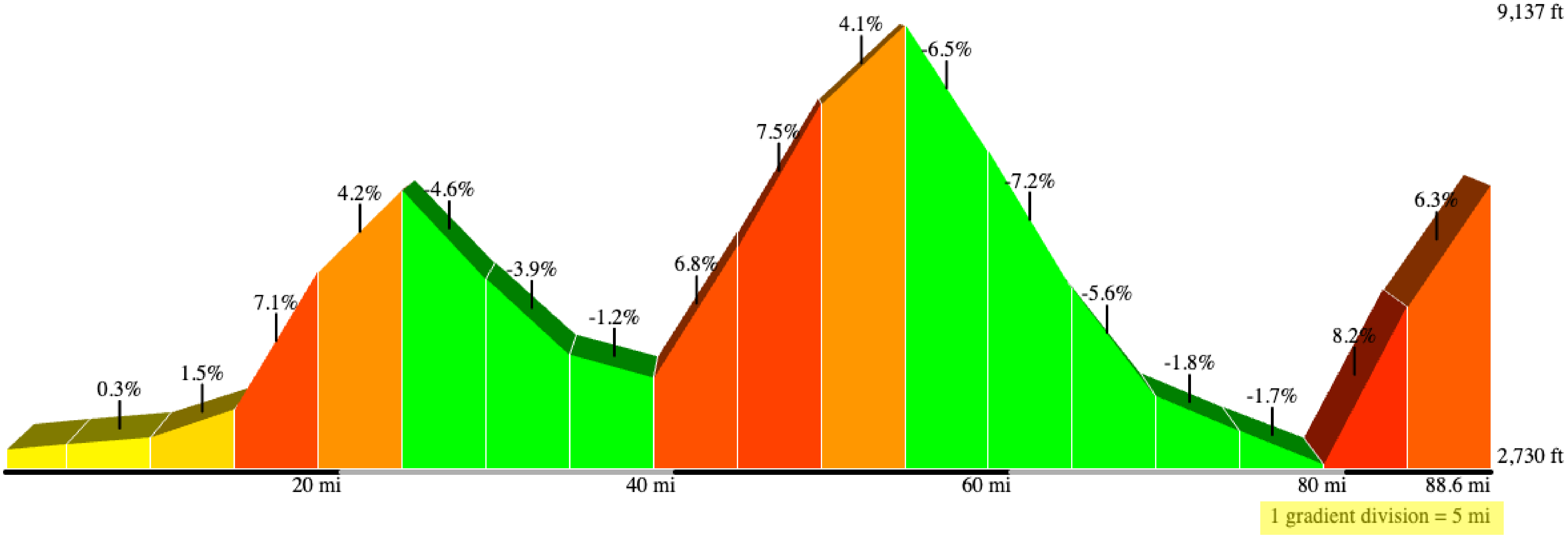
What part of the stage might be the most difficult? Why

Which part of the stage has the most difficult climbs

Which part of the stage will have the highest average speed?

Which part of the stage will have the lowest average speed?

What questions could you pose to students?



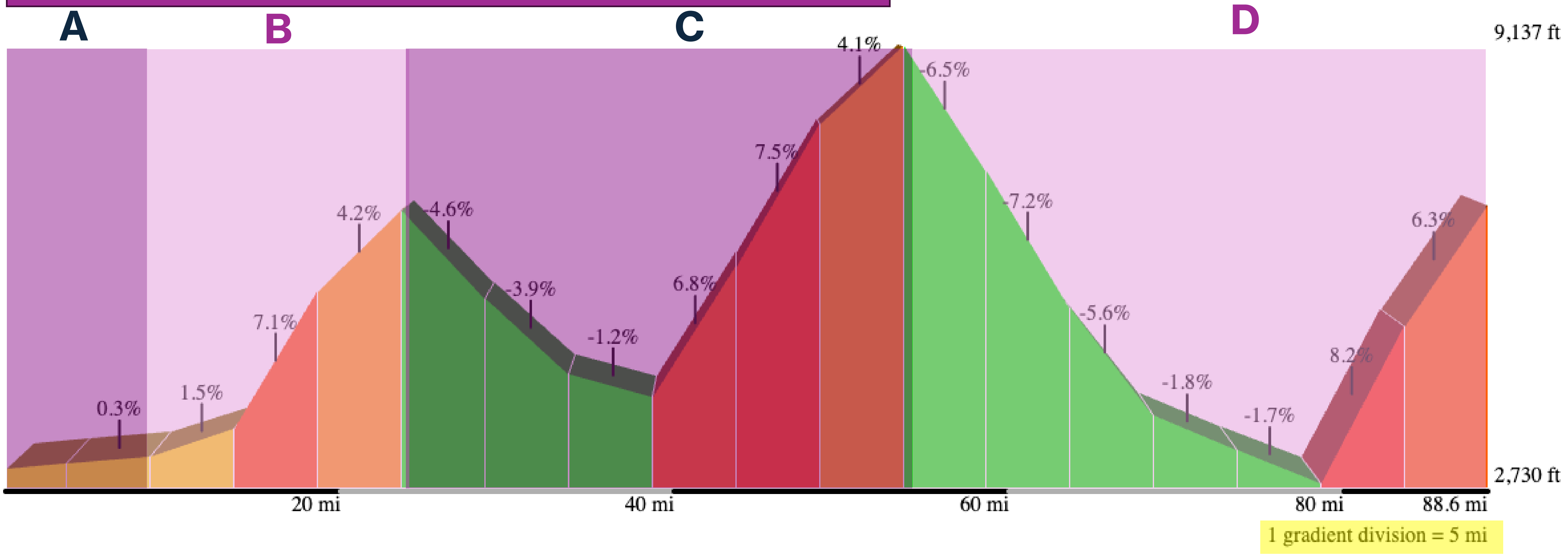
What part of the stage might be the most difficult? Why

Which part of the stage has the most difficult climbs

Which part of the stage will have the highest average speed?

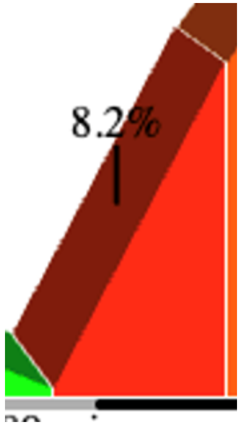
Which part of the stage will have the lowest average speed?

Let's explore out questions from earlier.. What do you think now



1 gradient division = 5 mi

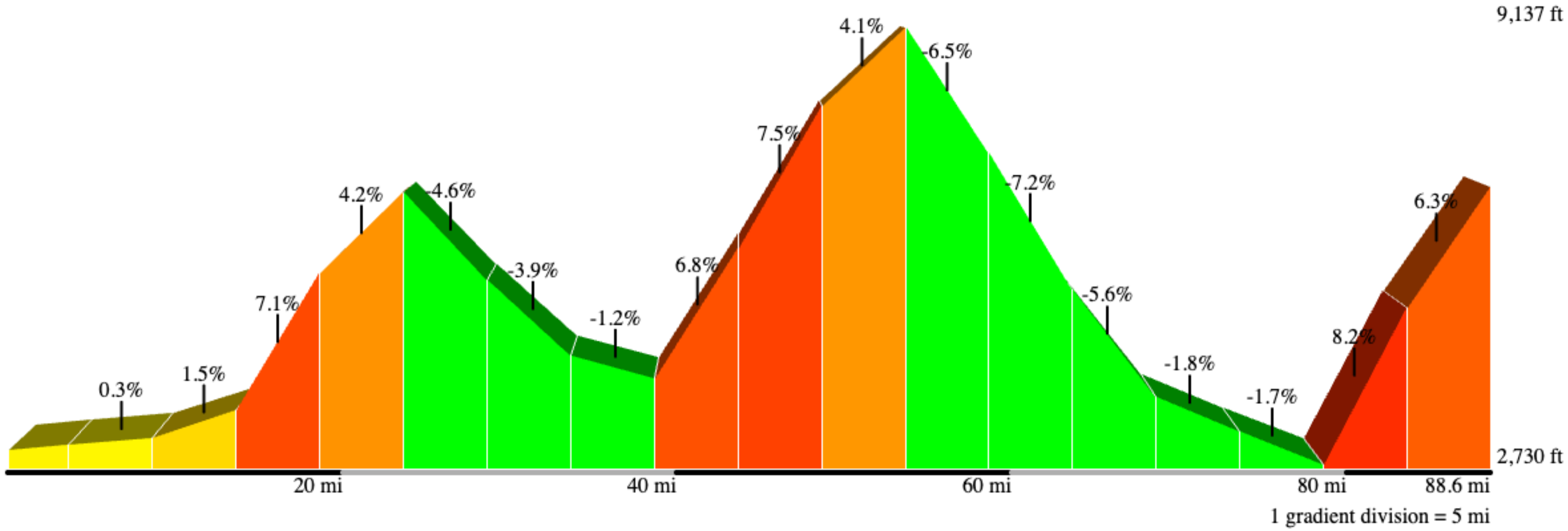
What might this mean?



How does this part of the stage compare

How does this part of the stage compare

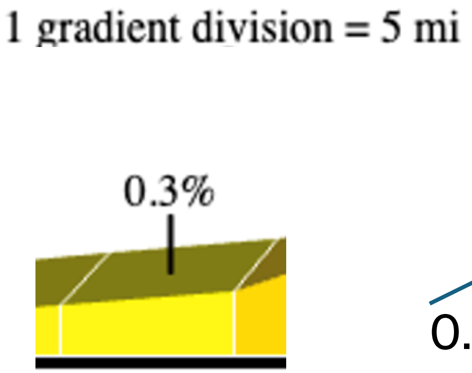
Why do you think it is a negative?



Category 1 climb

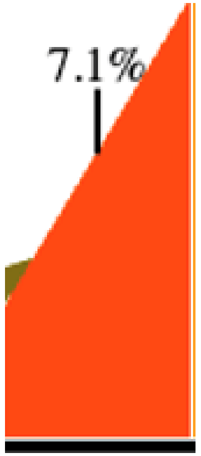
This climb averages 0.8%. The steepest quarter mile of this climb is 19.4% and steepest continuous mile is 11.3%. 3.6 miles of the climb is at or above 10% grade.
The gradient on this climb is broken down as follows:

- 42.1 miles (47.6%) of descent;
- 18.3 miles (20.7%) at 0-5% grade;
- 24.5 miles (27.7%) at 5-10% grade;
- 2.9 miles (3.3%) at 10-15% grade;
- 0.5 miles (0.6%) at 15-20% grade;
- 0.2 miles (0.2%) at 20%+ grade



0.3% rise in height (measured in feet)

5 miles horizontally

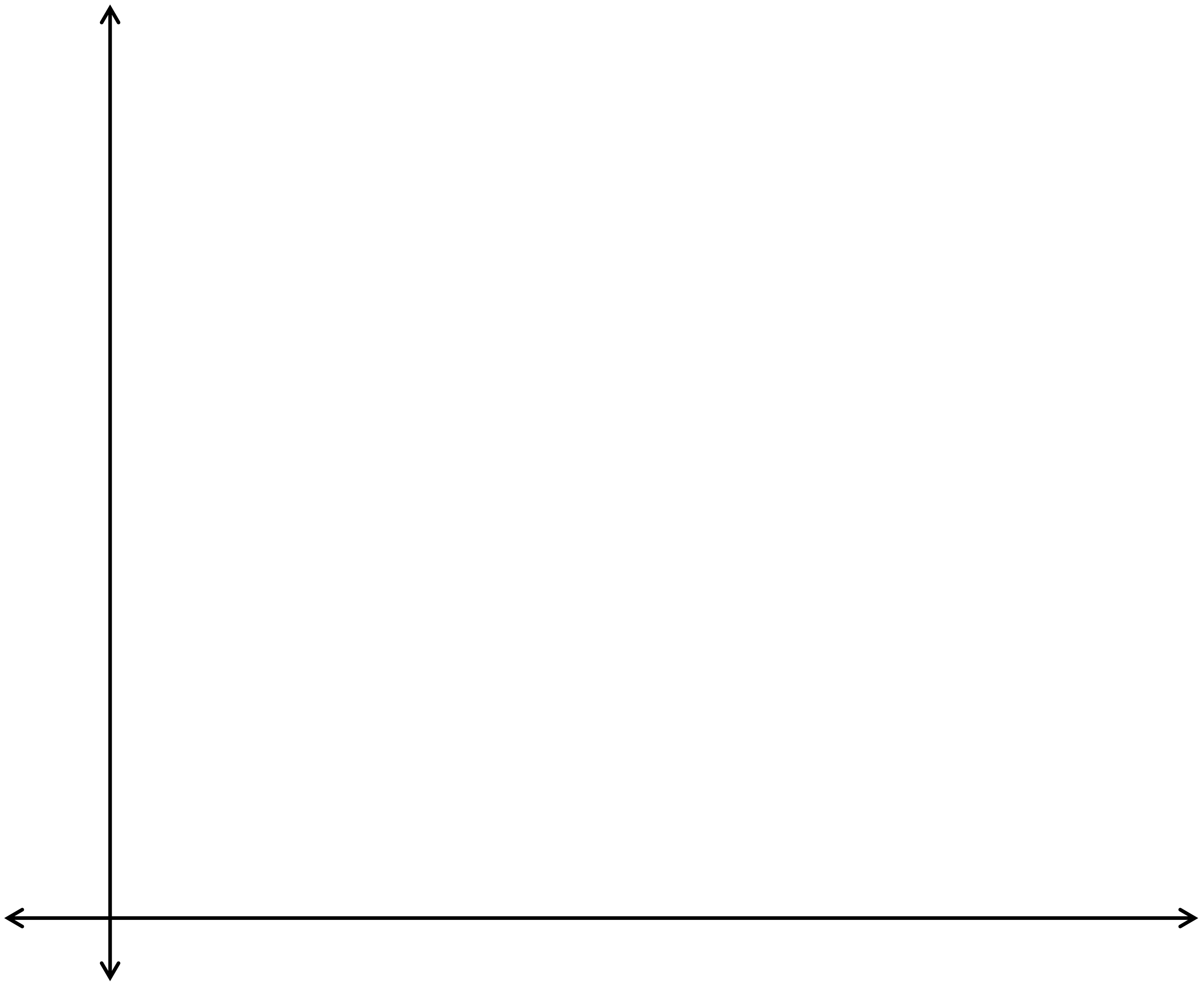


7.1% rise in height (measured in feet)

How could we describe this piece of the stage

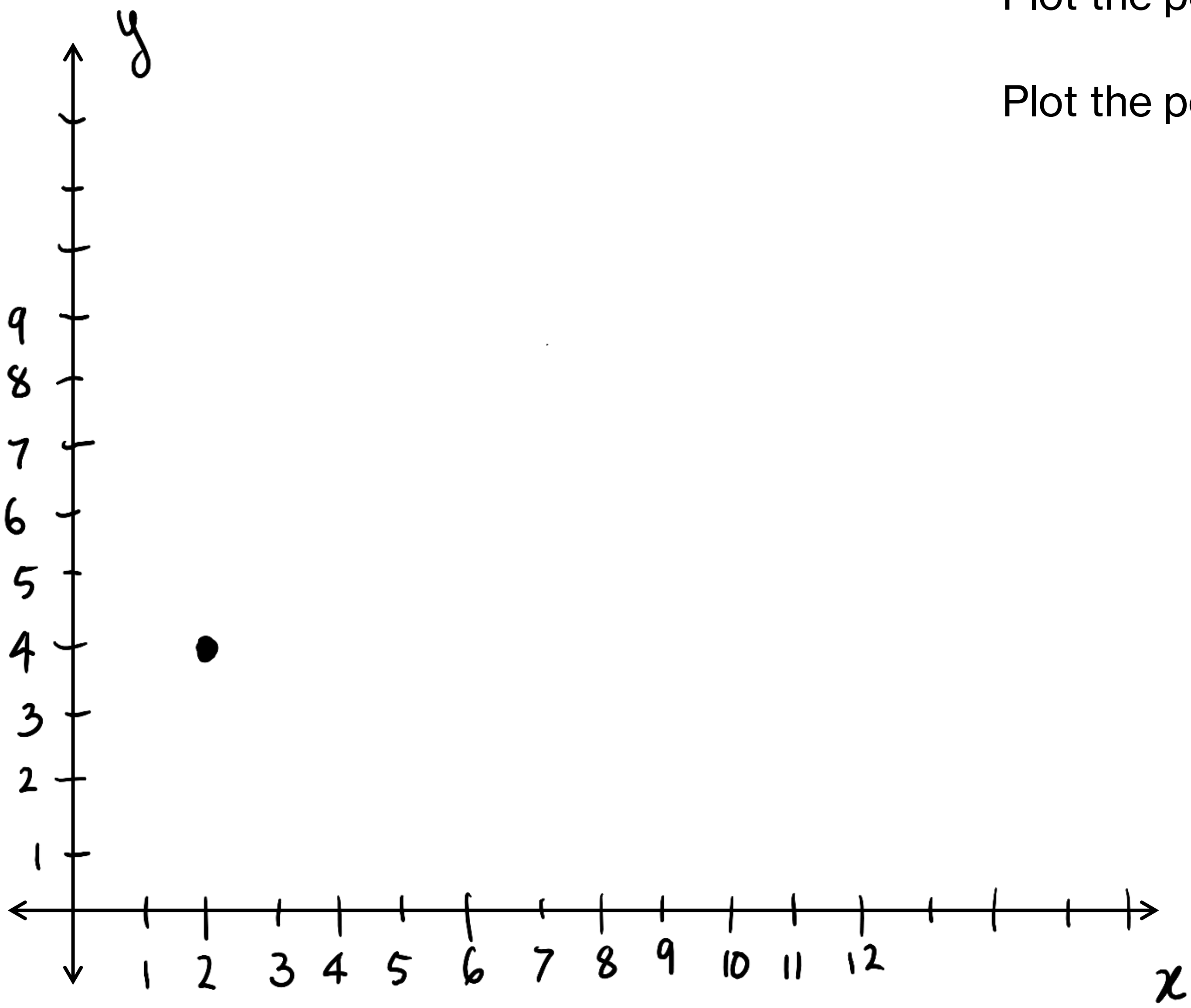
5 miles horizontally

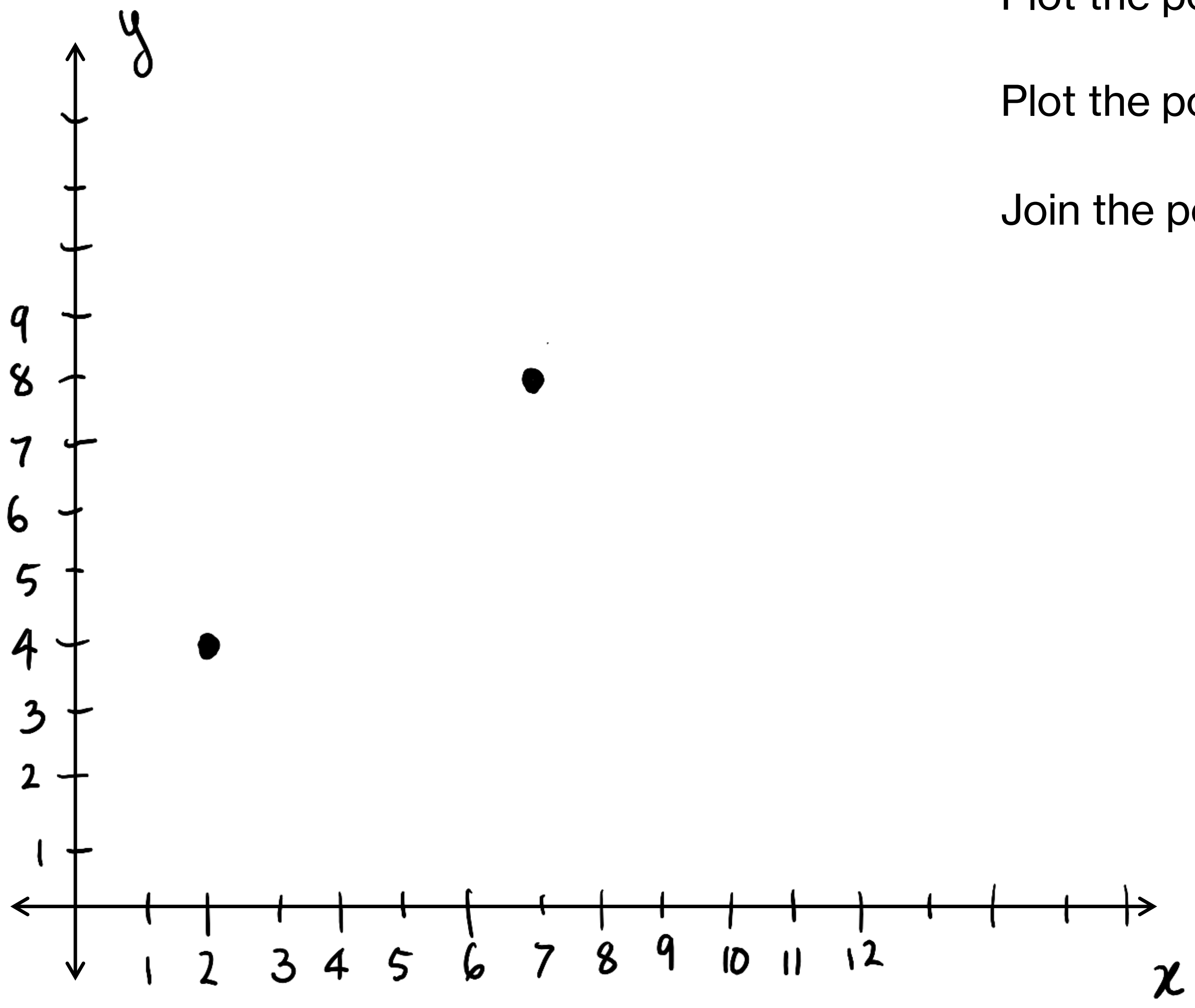
Plot the point (2, 4)



Plot the point (2, 4)

Plot the point (7, 8)

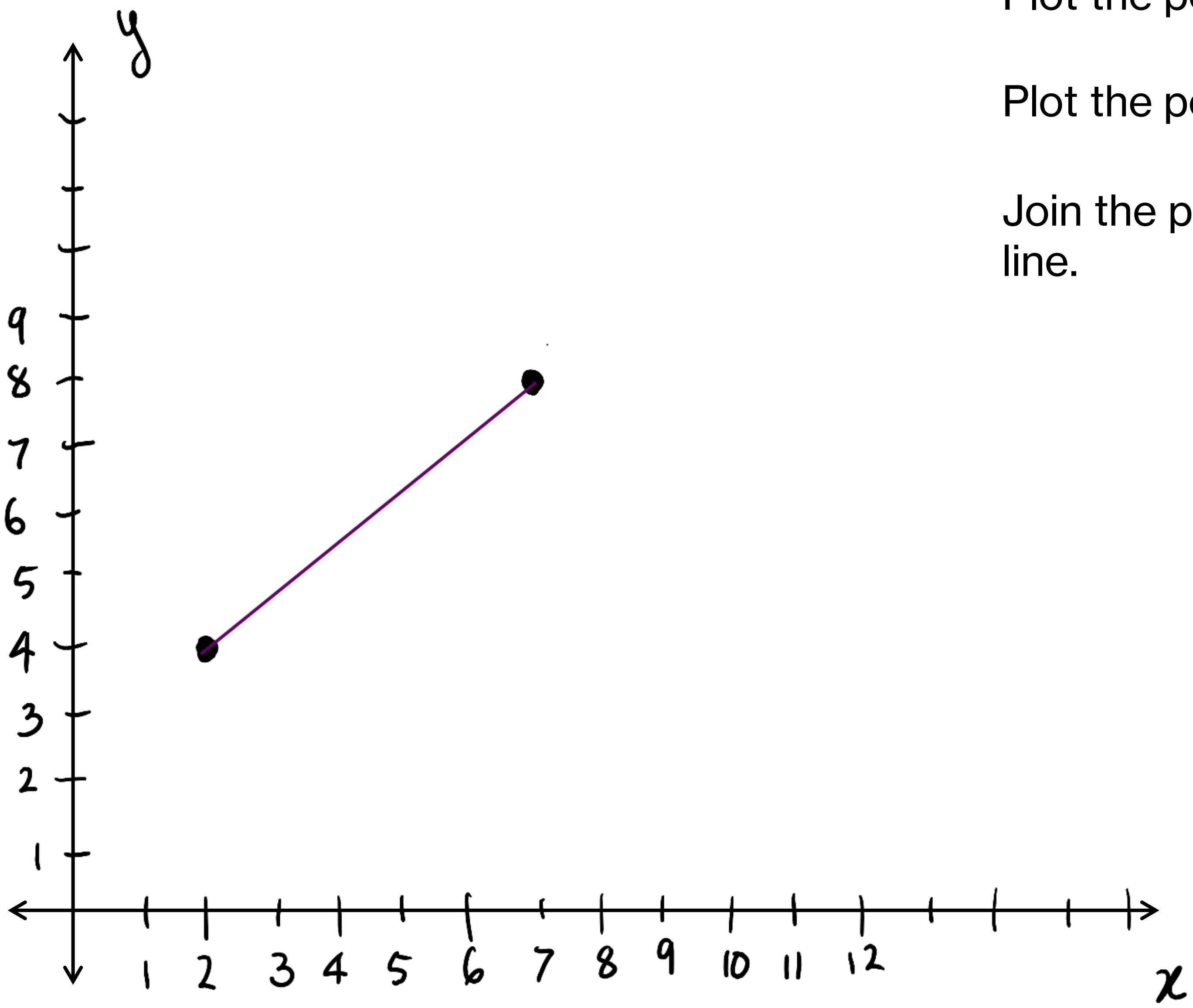




Plot the point (2, 4)

Plot the point (7, 8)

Join the points with a line.



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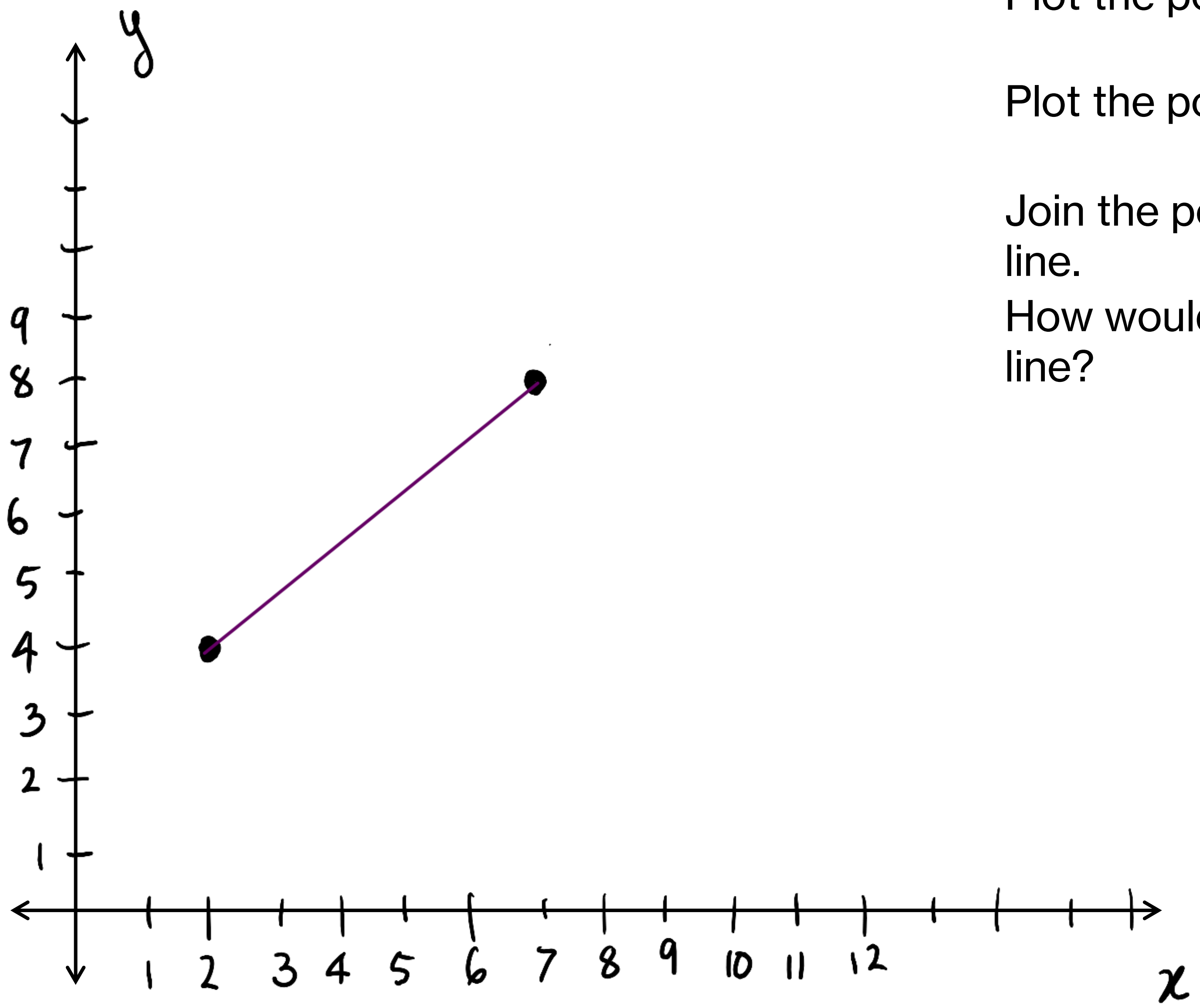

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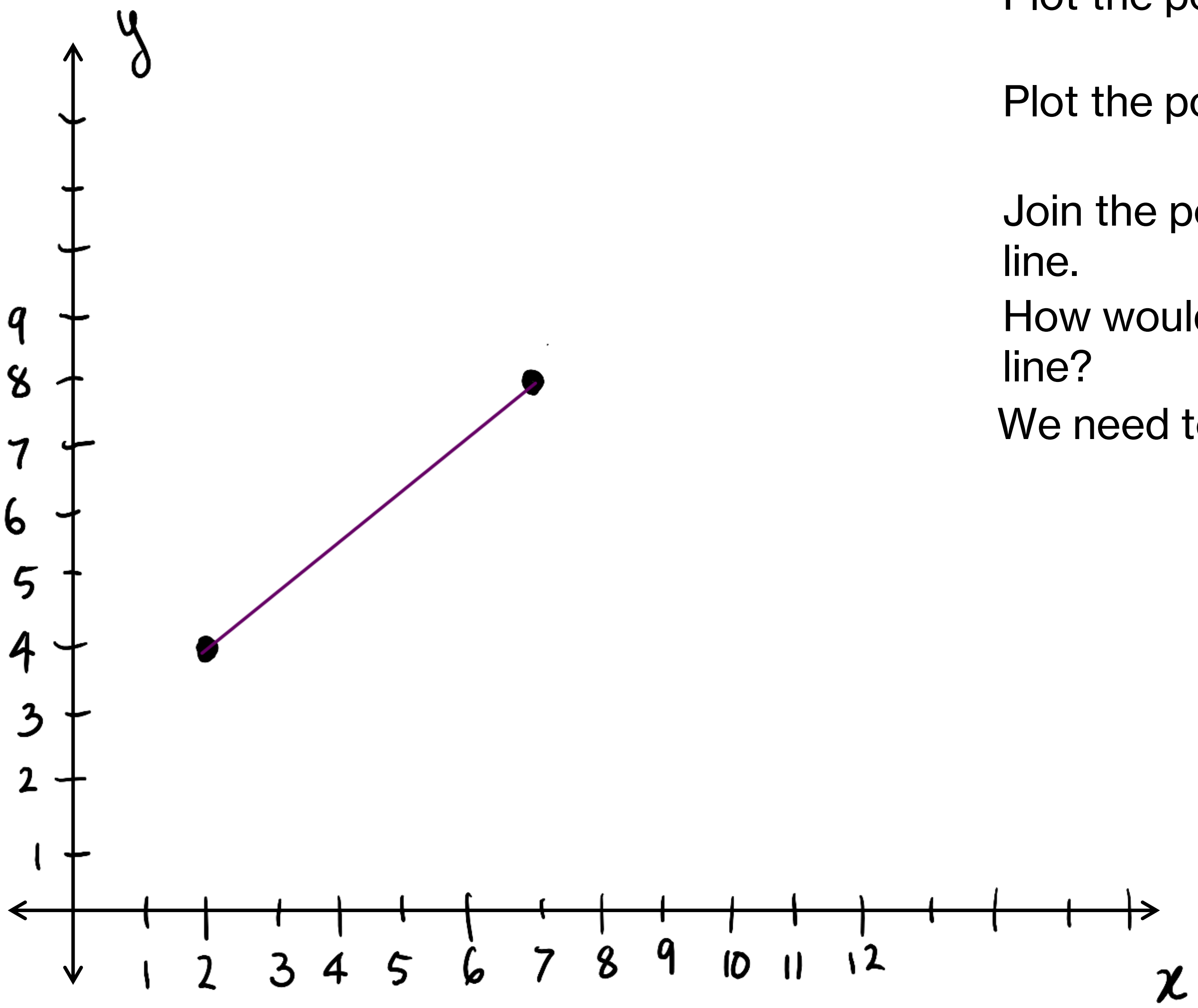
Join the points with a line.

How would we find the slope of the line?



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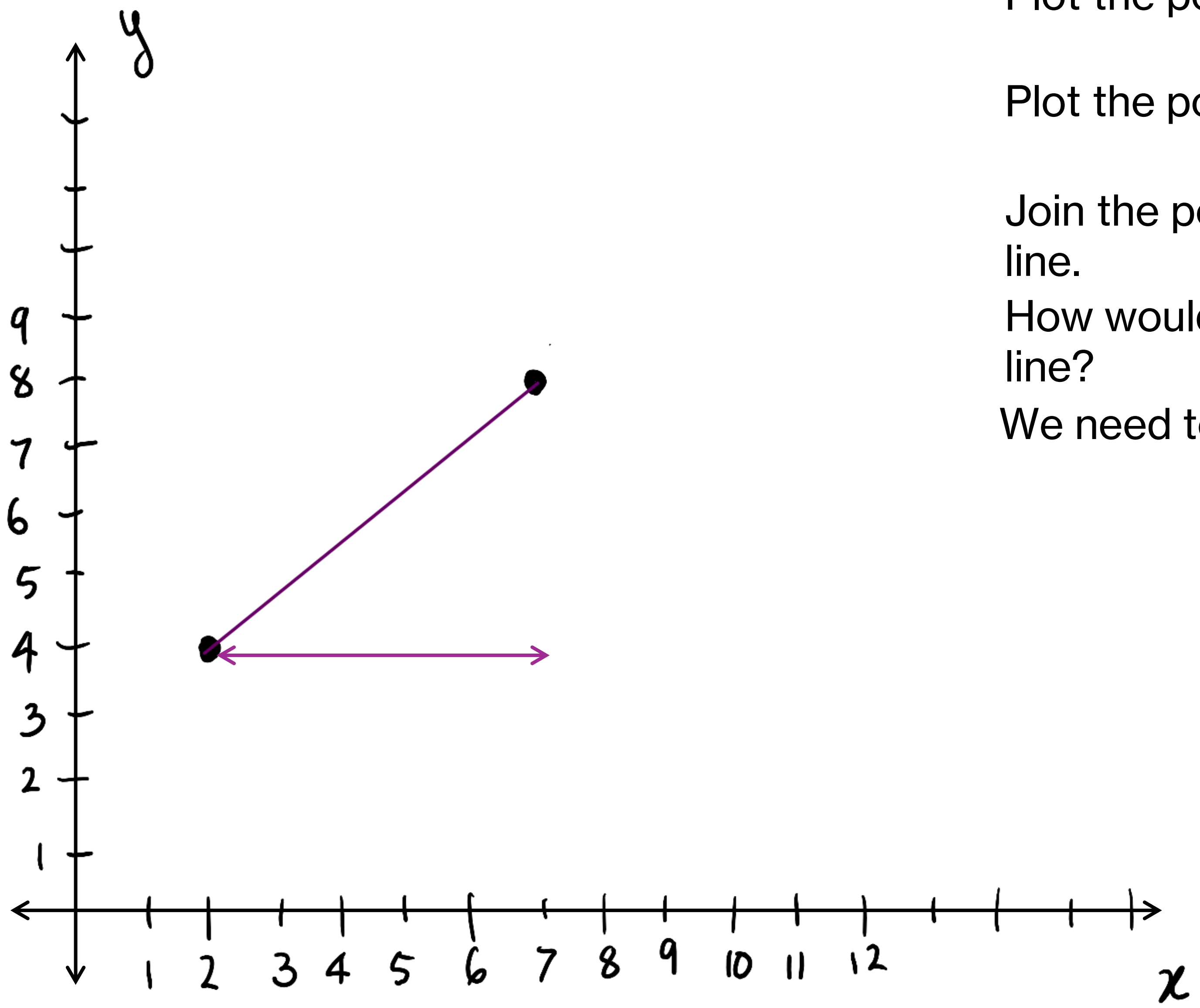
How would we find the slope of the line?

We need to know the horizontal distance...



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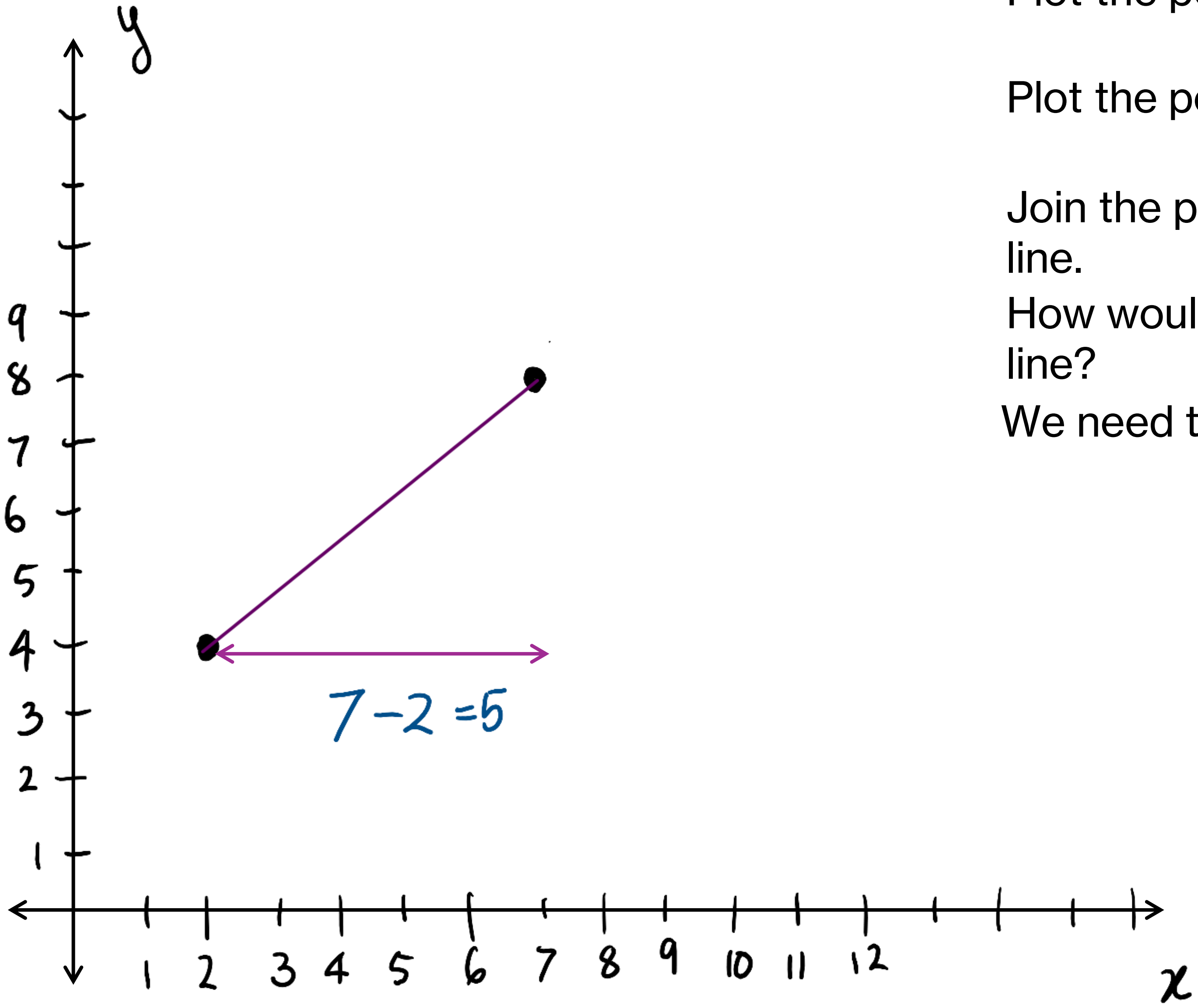

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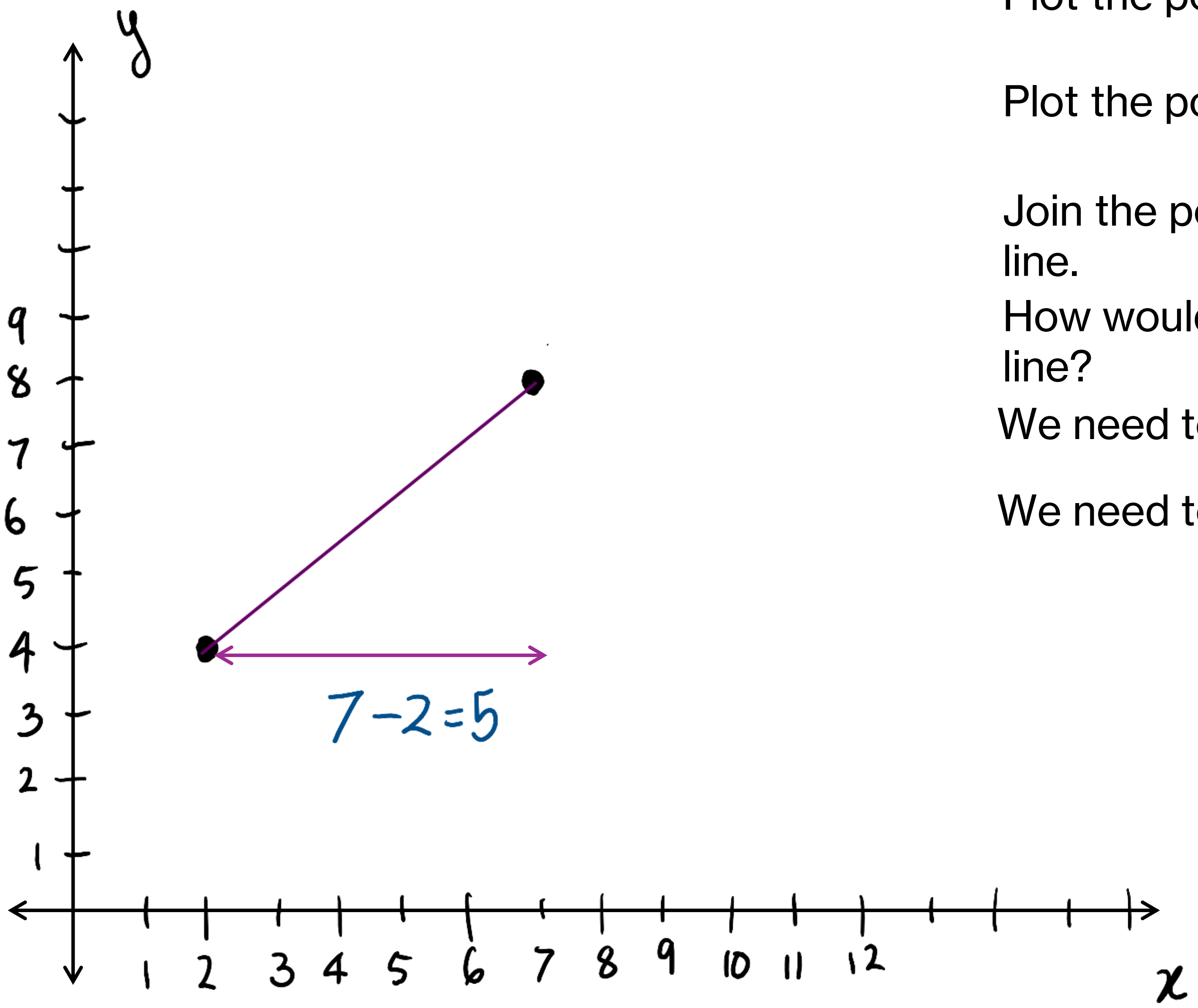

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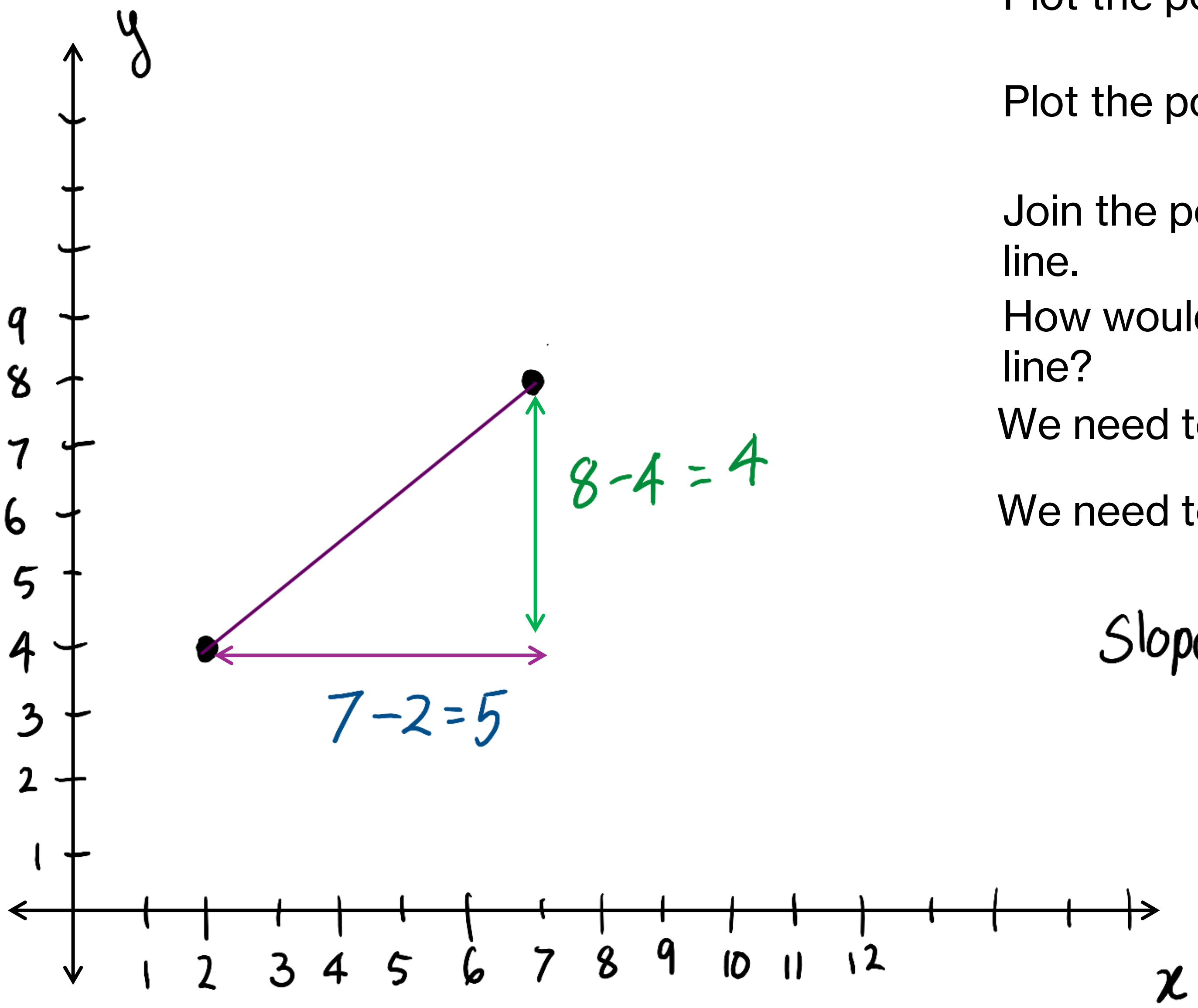
How would we find the slope of the line?

We need to know the horizontal distance...

We need to know the vertical distance...



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Plot the point (7, 8)

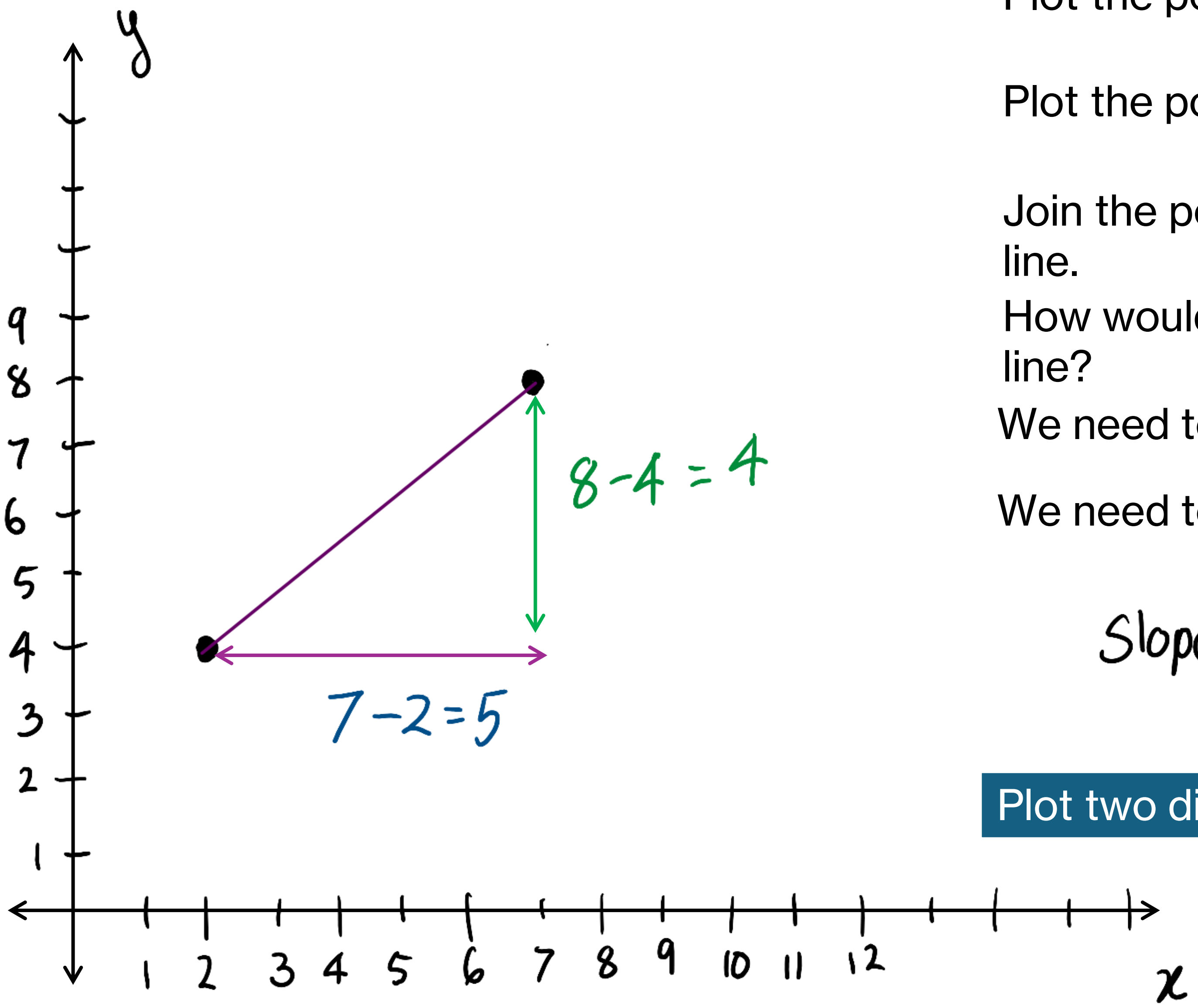
Join the points with a line.

How would we find the slope of the line?

We need to know the horizontal distance...

We need to know the vertical distance...

$$\text{Slope} = \frac{4}{5}$$



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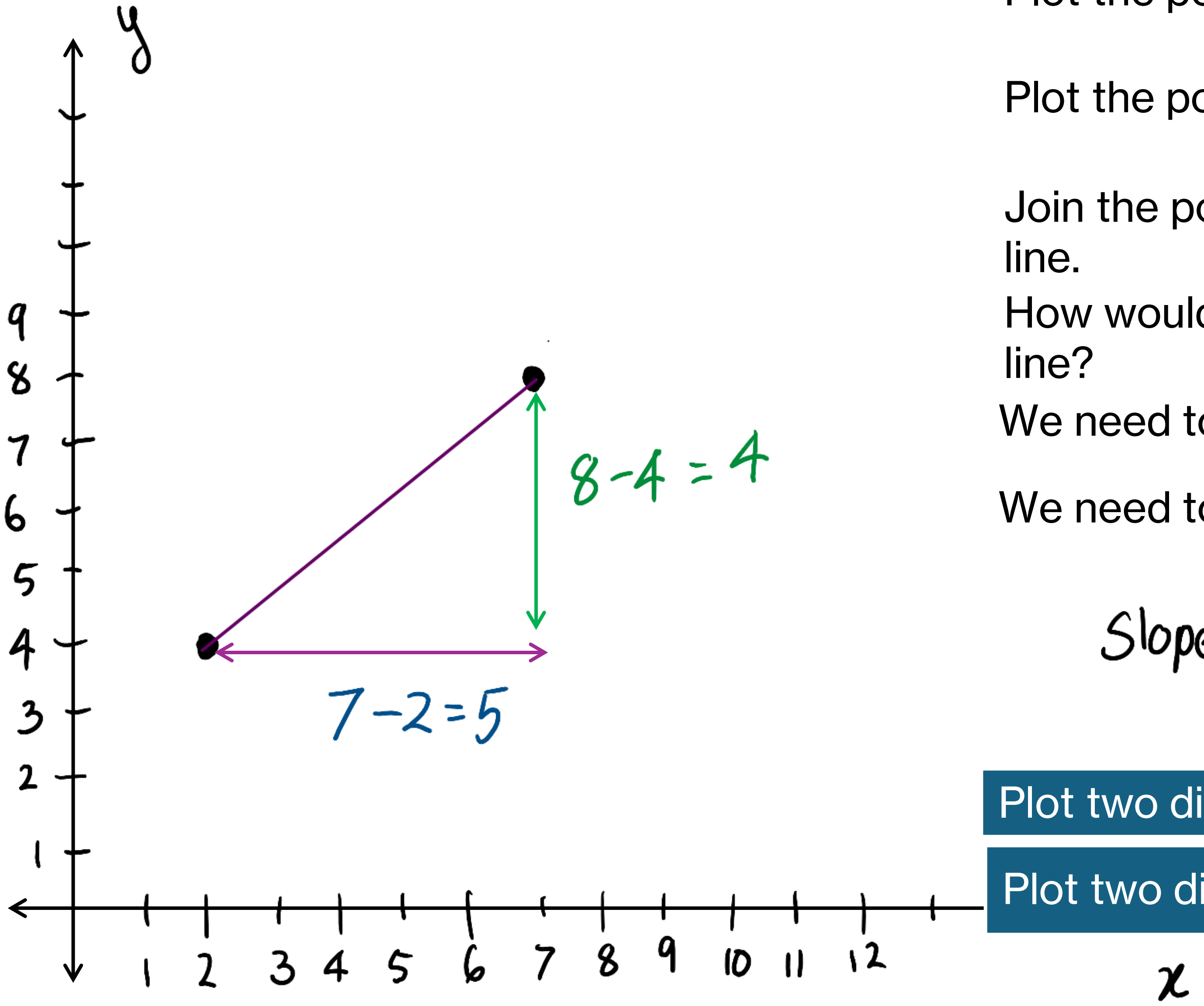
$$\text{Slope} = \frac{4}{5}$$

Plot two different points that would have the same slop



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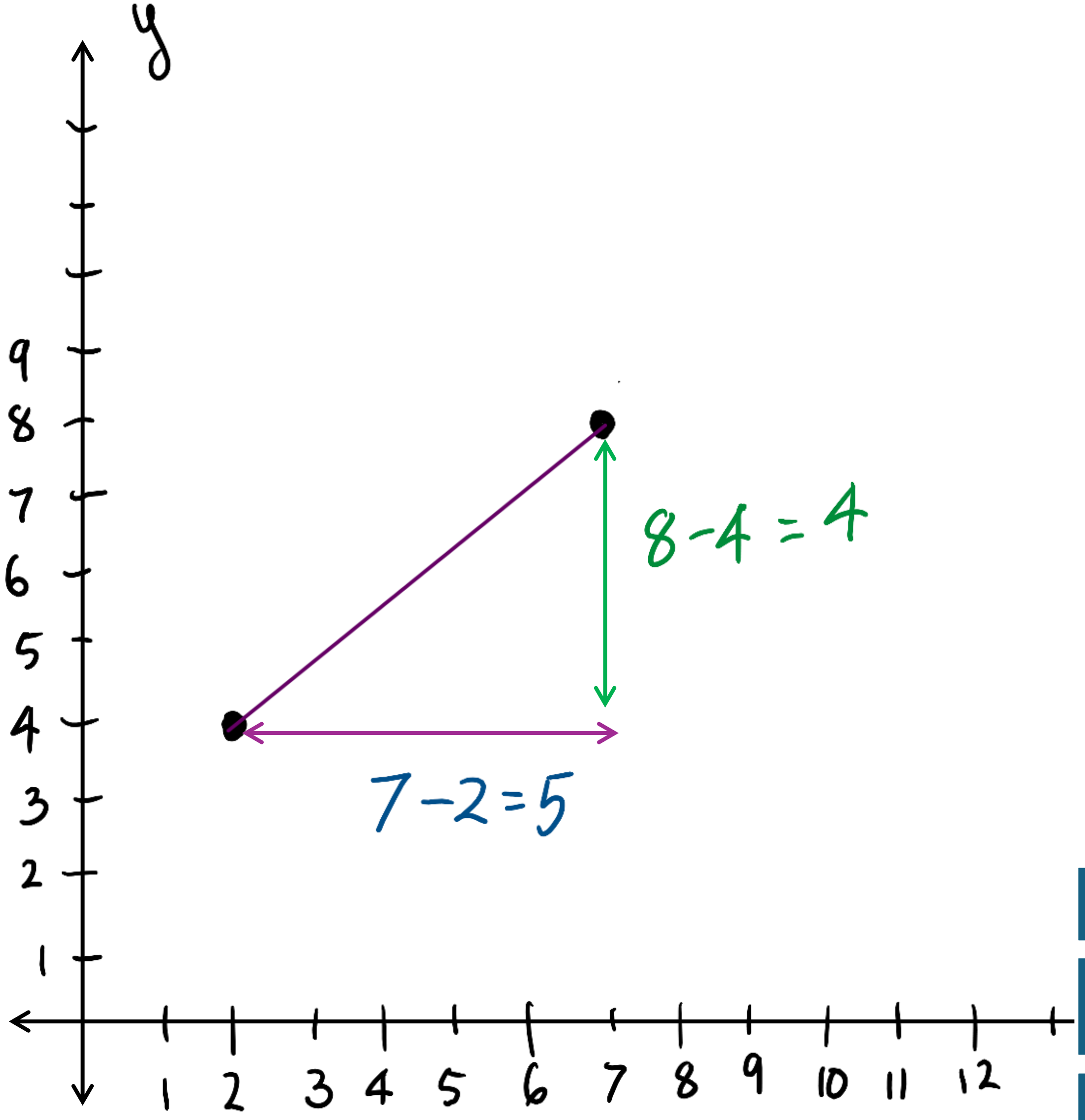
$$\text{Slope} = \frac{4}{5}$$

Plot two different points that would have the same slop

Plot two different points that would have a slope of –



Plot the point (2, 4)

Plot the point (7, 8)

Join the points with a line.

How would we find the slope of the line?

We need to know the horizontal distance...

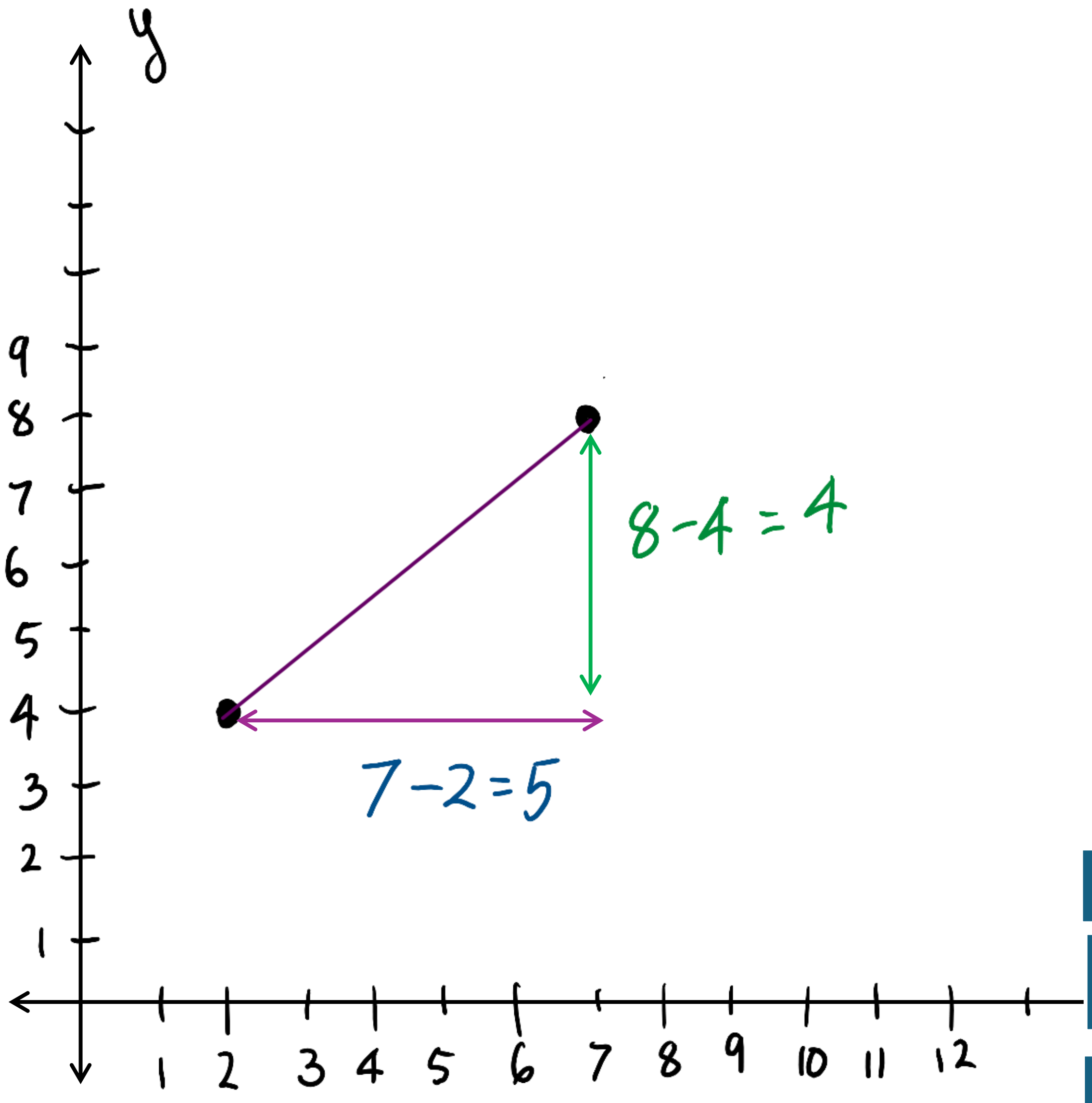
We need to know the vertical distance...

$$\text{Slope} = \frac{4}{5}$$

Plot two different points that would have the same slop

Plot two different points that would have a slope of –

Develop a formula for finding the slope between two



Plot the point (2, 4)

Plot the point (7, 8)

Join the points with a line.

How would we find the slope of the line?

We need to know the horizontal distance...

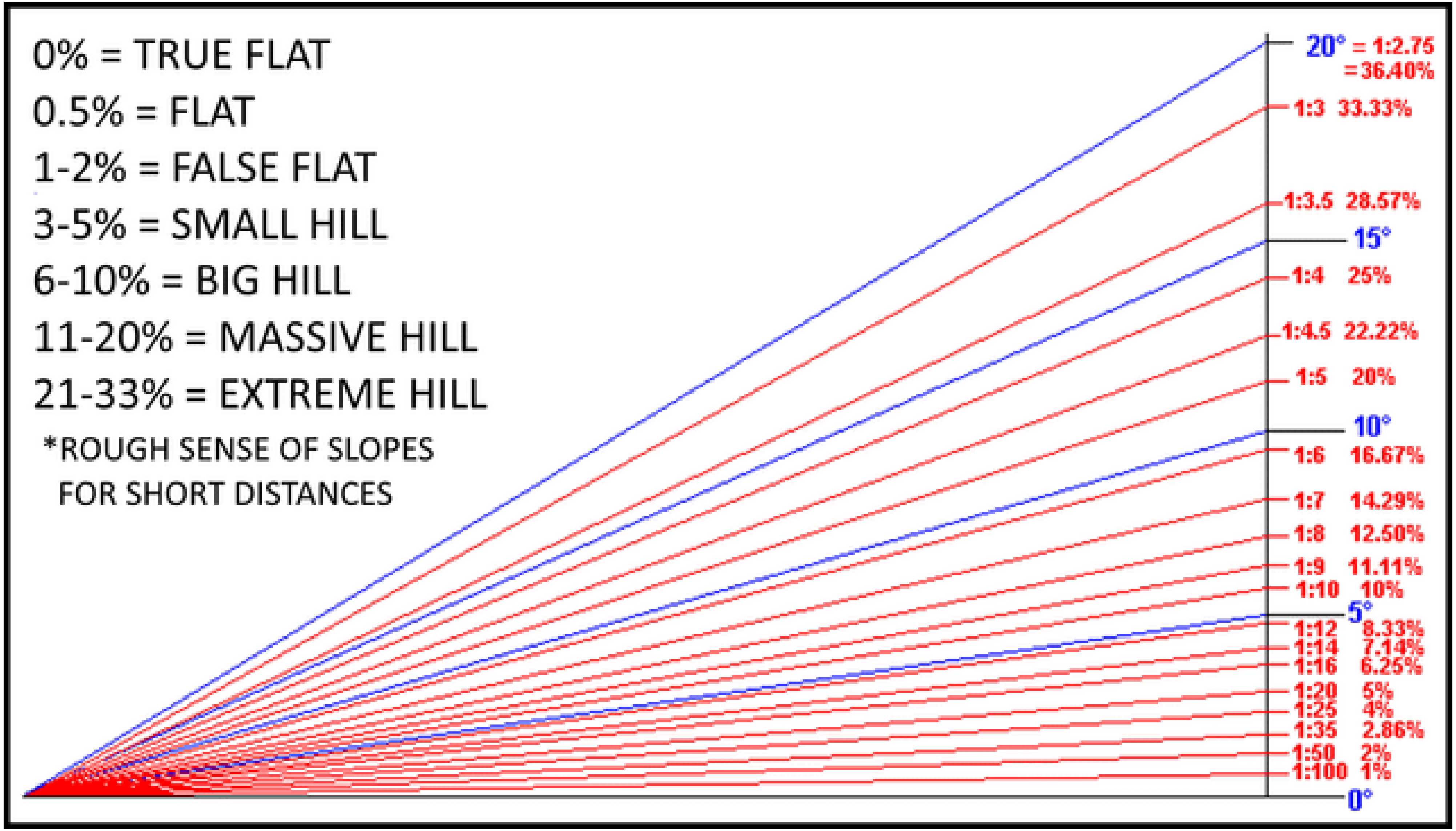
We need to know the vertical distance...

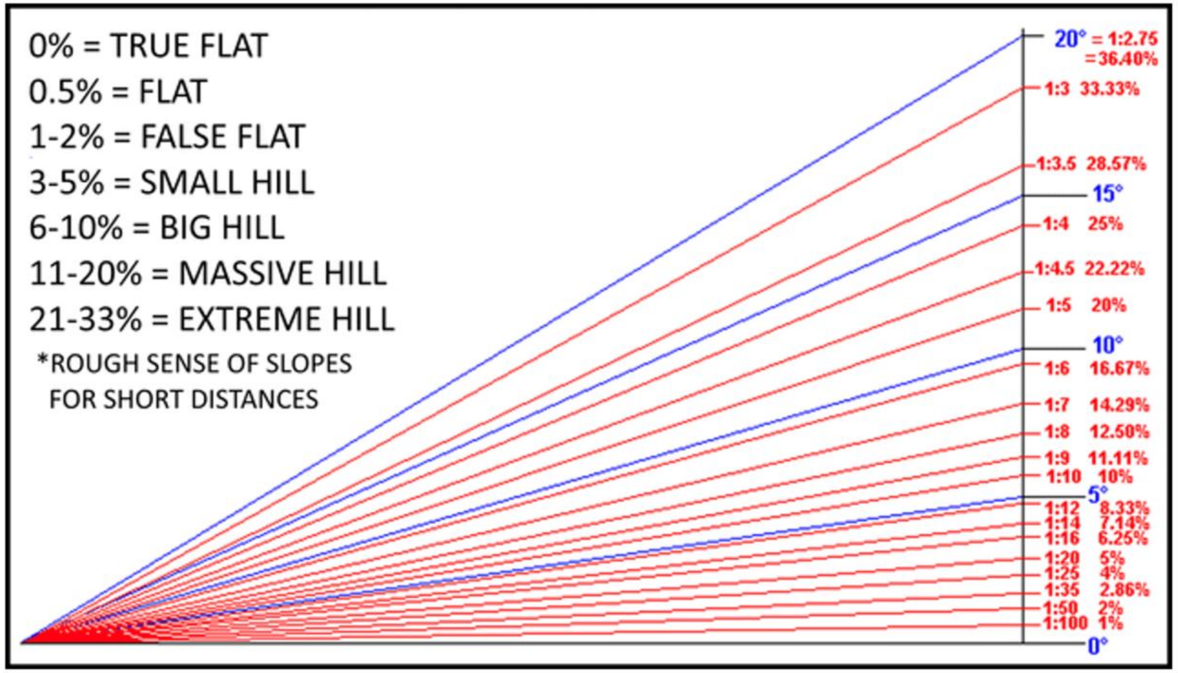
Slope = $\frac{4}{5}$

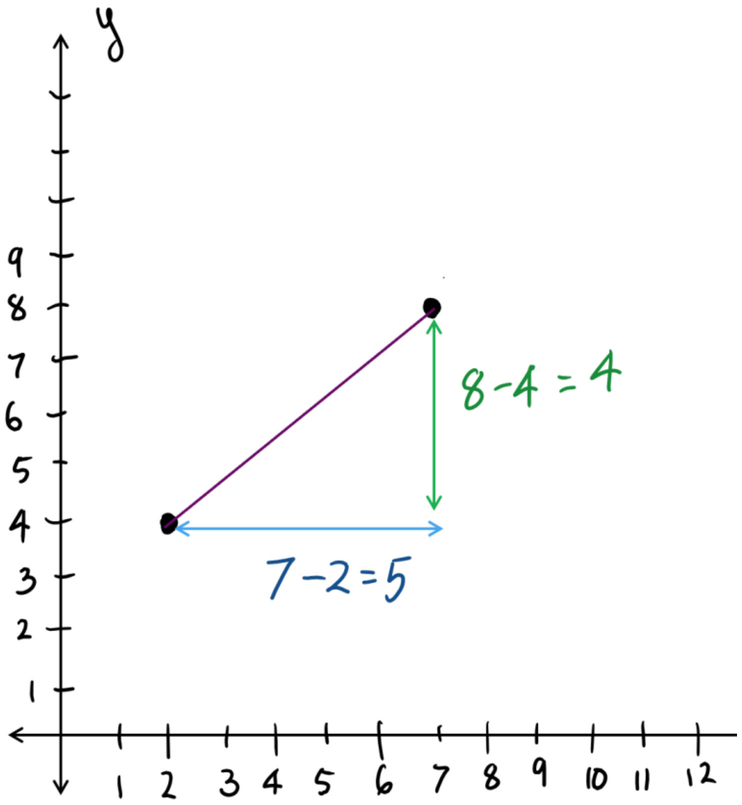
Plot two different points that would have the same slope

Plot two different points that would have a slope of $-\frac{4}{5}$

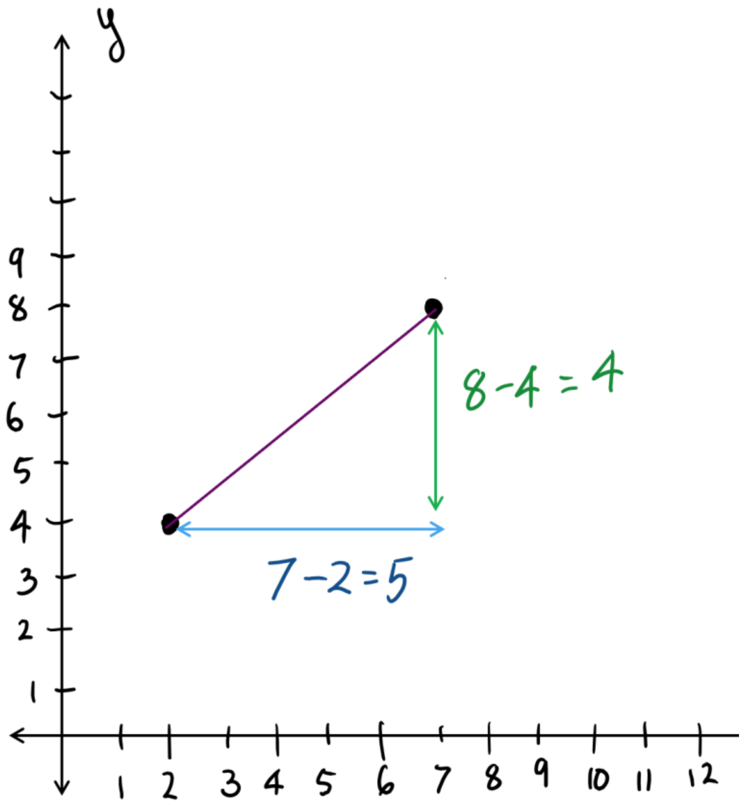
Calculate the slope of $\frac{4}{5}$ as percentage


0% = TRUE FLAT
0.5% = FLAT
1-2% = FALSE FLAT
3-5% = SMALL HILL
6-10% = BIG HILL
11-20% = MASSIVE HILL
21-33% = EXTREME HILL
*ROUGH SENSE OF SLOPES
FOR SHORT DISTANCES
20° = 1:2.75
=36.40%
1:3 33.33%
1:3.5 28.57%
15°
1:4 25%
1:4.5 22.22%
1:5 20%
10°
1:6 16.67%
1:7 14.29%
1:8 12.50%
1:9 11.11%
1:10 10%
5°
1:12 8.33%
1:14 7.14%
1:16 6.25%
1:20 5%
1:25 4%
1:35 2.86%
1:50 2%
1:100 1%
0°

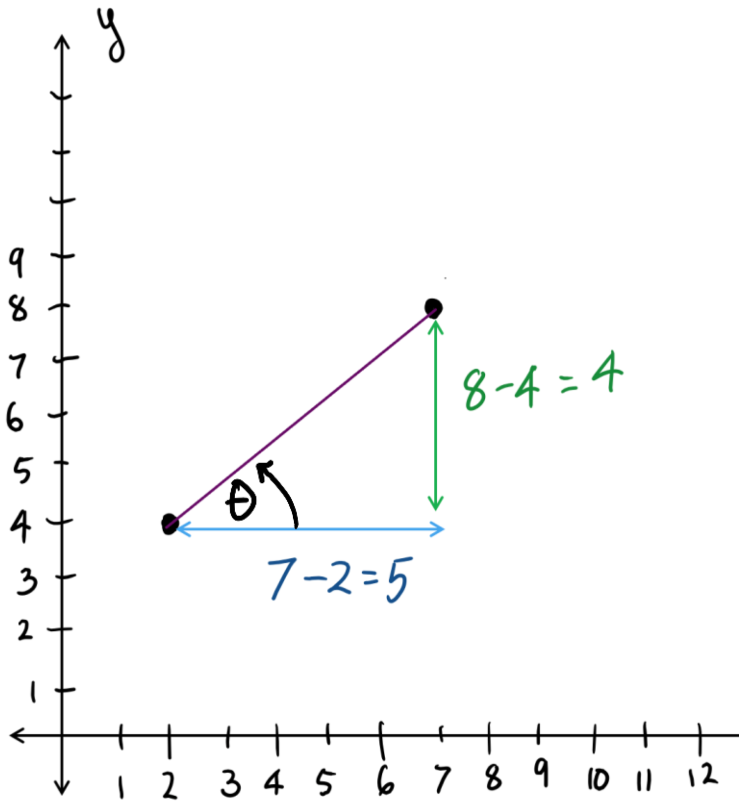




Would a slope like this be realistic in the Tour de France



Find the angle of this gradient?



Find the angle of this gradient?

$$\tan(\theta) = \frac{4}{5}$$

$$\theta = \tan^{-1}\left(\frac{4}{5}\right)$$

$$\theta \approx 38.67^{\circ}$$

BREAK

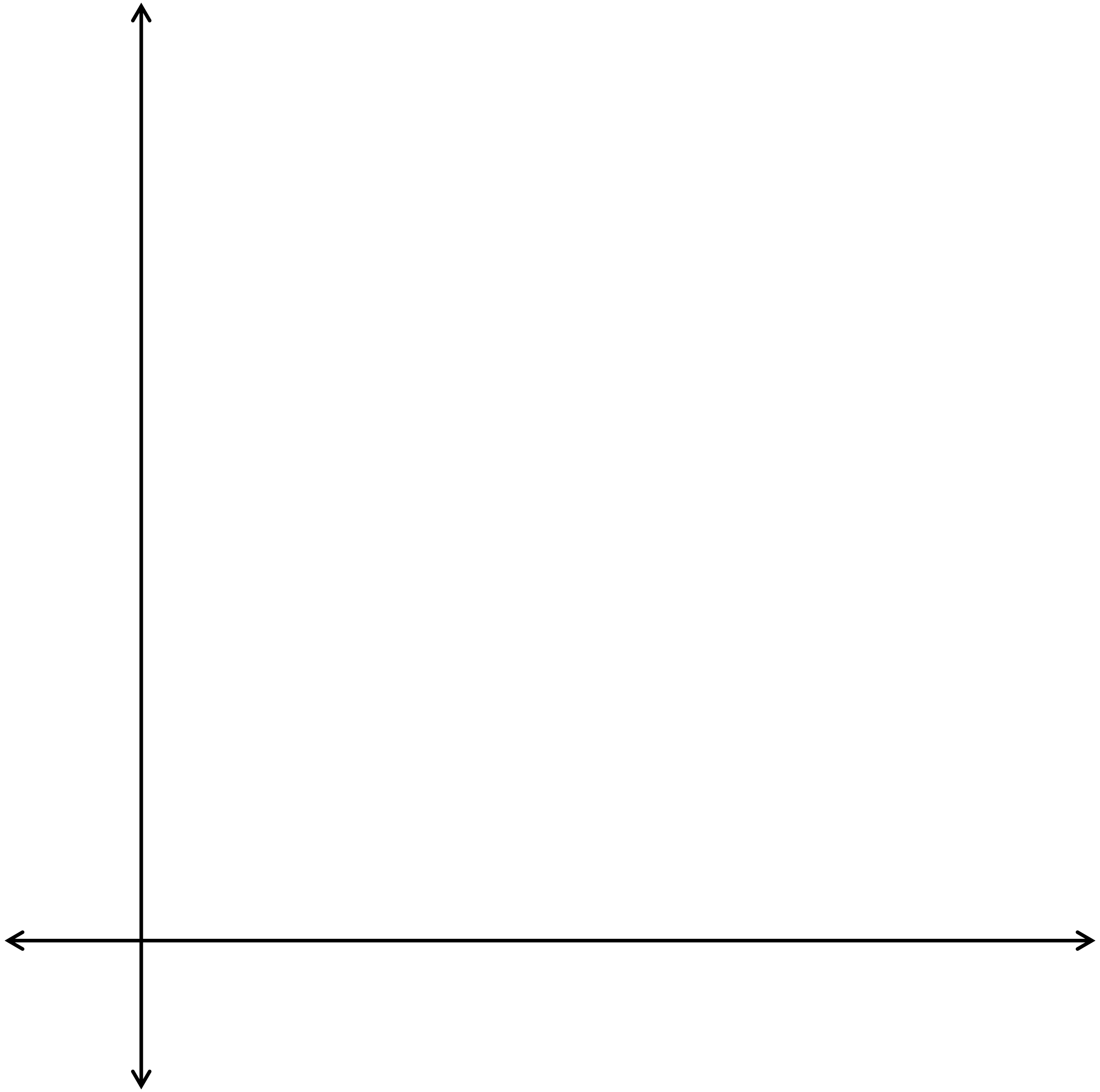
15 minutes

Creating and seizing opportunities

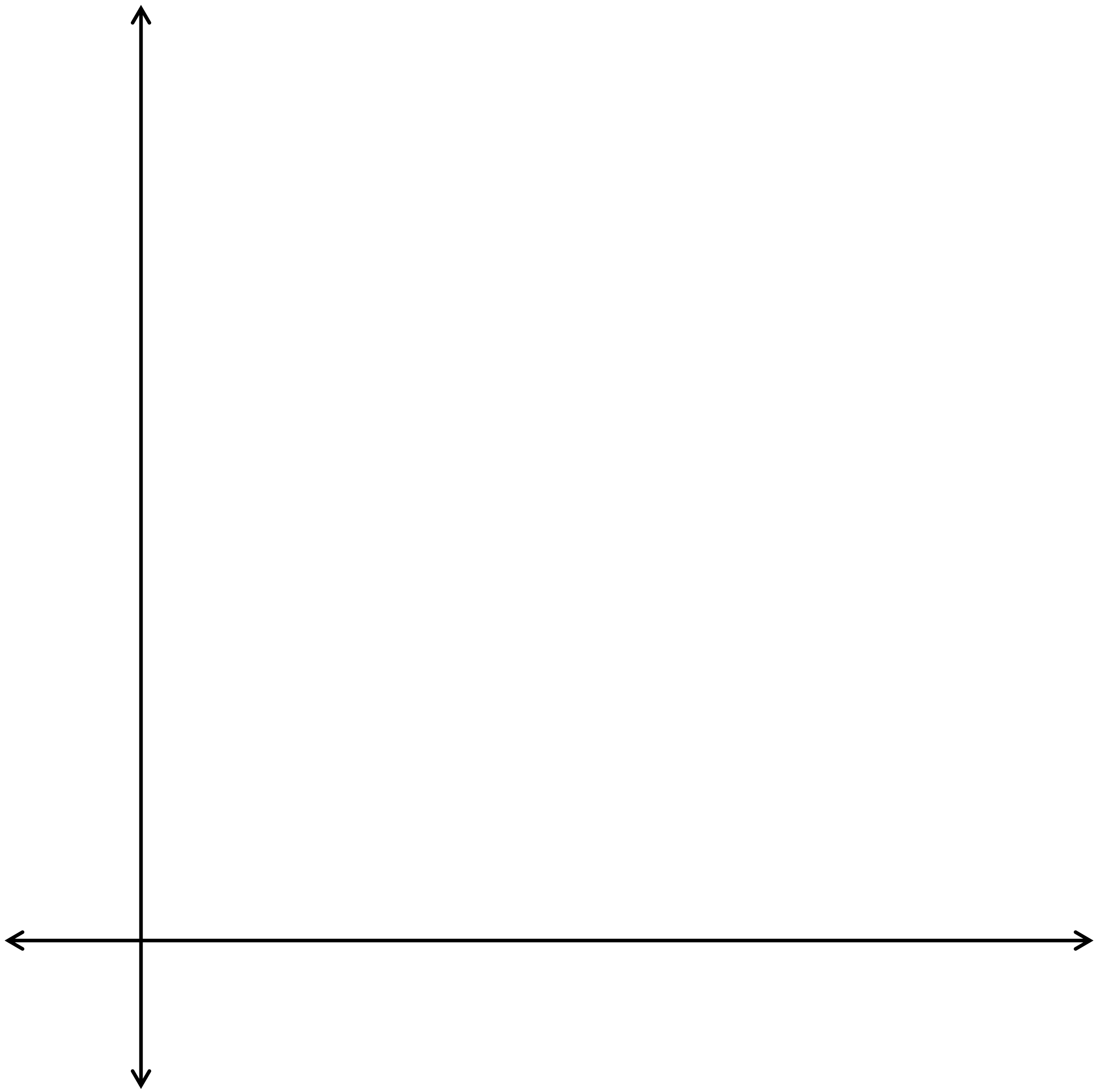
Guiding towards how *and why...*

Discuss approaches to teaching…

The product of the slopes of two perpendicular lines is -1.

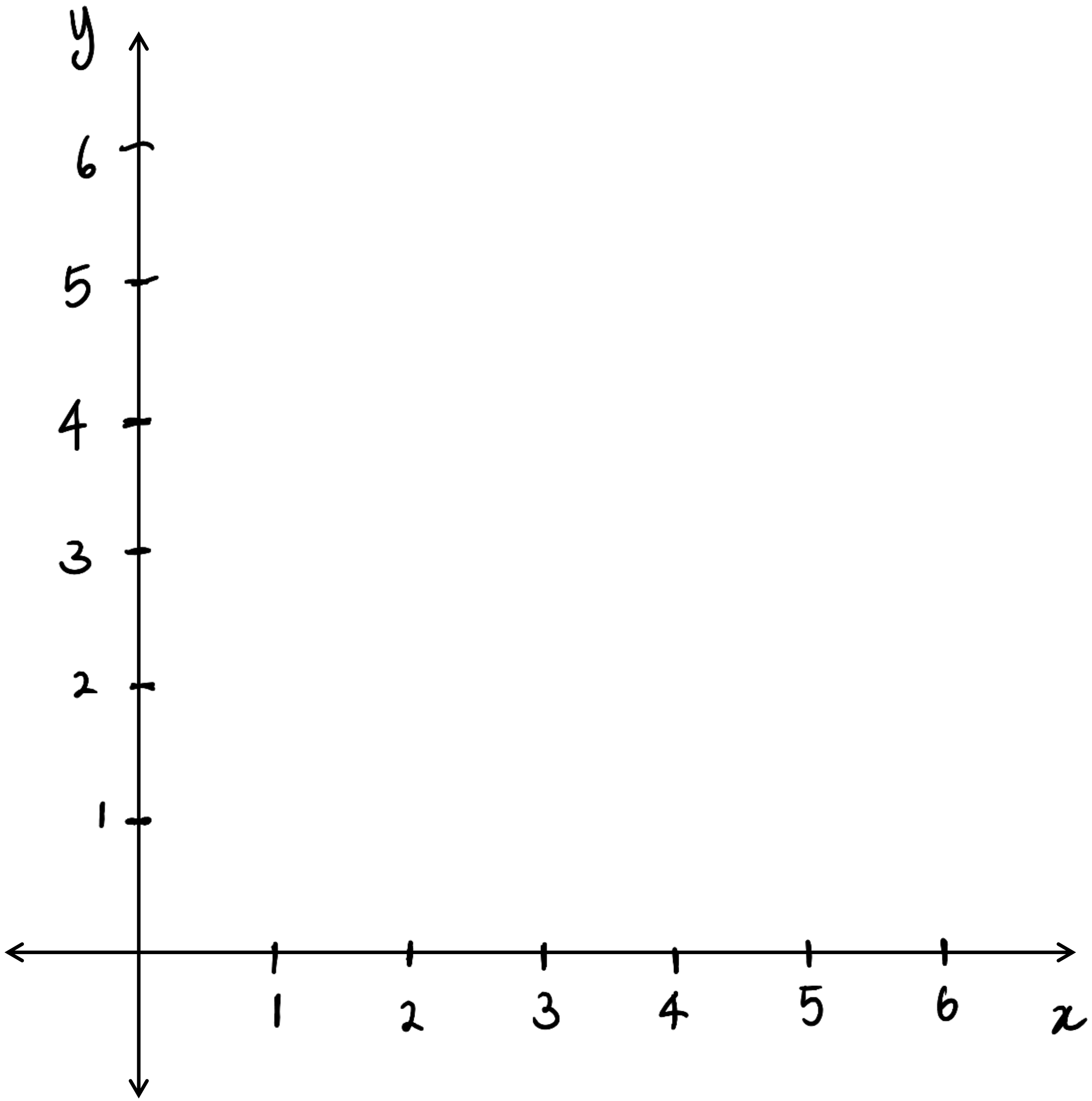


Plot two points so the angle of the slope is 30°



Plot two points so the angle of the slope is 30°

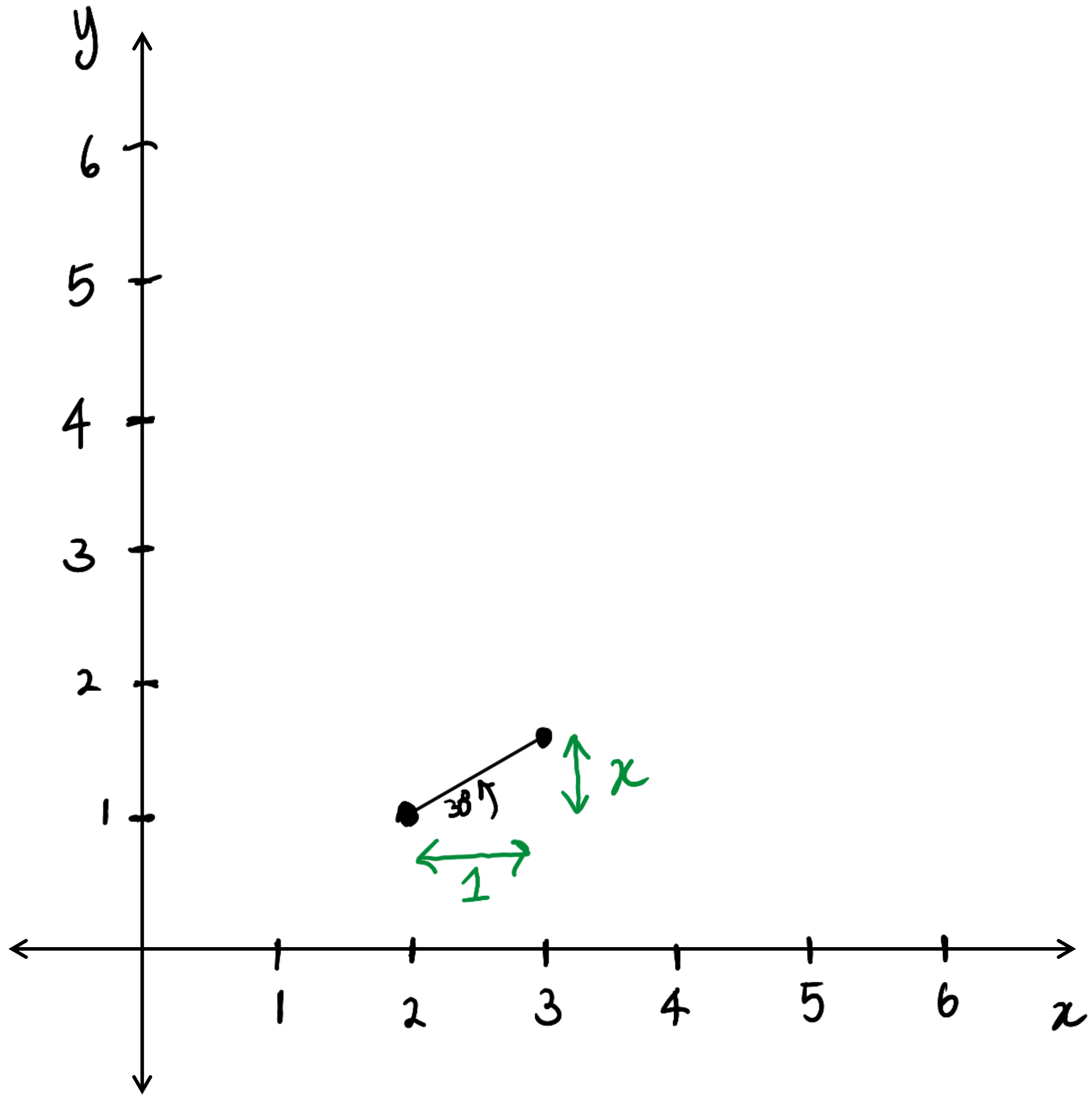
Hint: $\tan(30°) = \frac{1}{\sqrt{3}}$



Plot two points so the angle of the slope is 30°

Hint: $\tan(30°) = \frac{1}{\sqrt{3}}$

$\tan(\theta) = \frac{o}{a}$



Plot two points so the angle of the slope is 30°

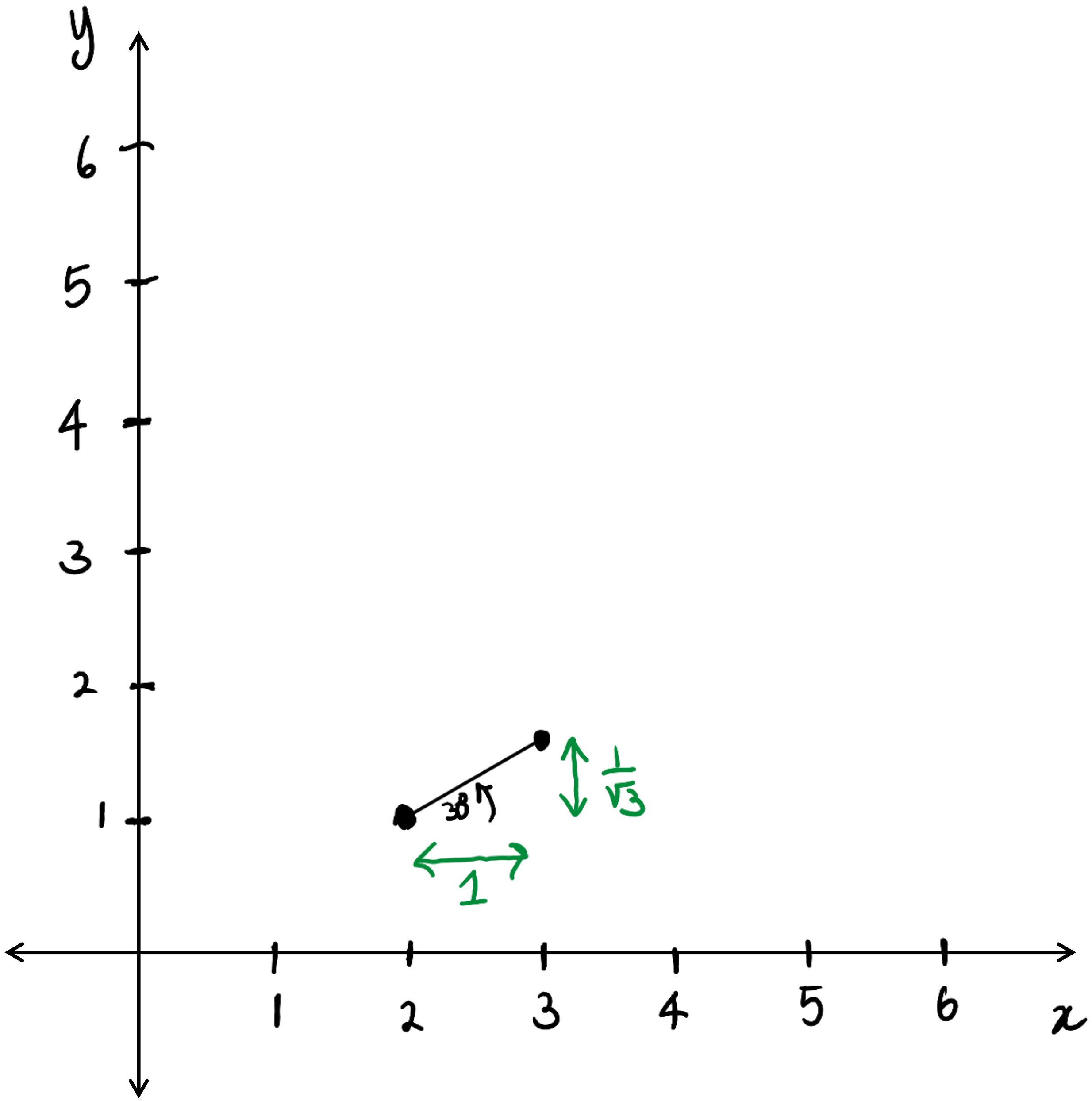
Hint: tan (30 °) = $\frac{1}{\sqrt{3}}$

$$\tan(\theta) = \frac{o}{a}$$

$$\tan(30^\circ) = \frac{1}{\sqrt{3}} = \frac{x}{1}$$

$$1 = \sqrt{3} \cdot x$$

$$\frac{1}{\sqrt{3}} = x$$



Plot two points so the angle of the slope is 30°

Hint: tan (30 °) = $\frac{1}{\sqrt{3}}$

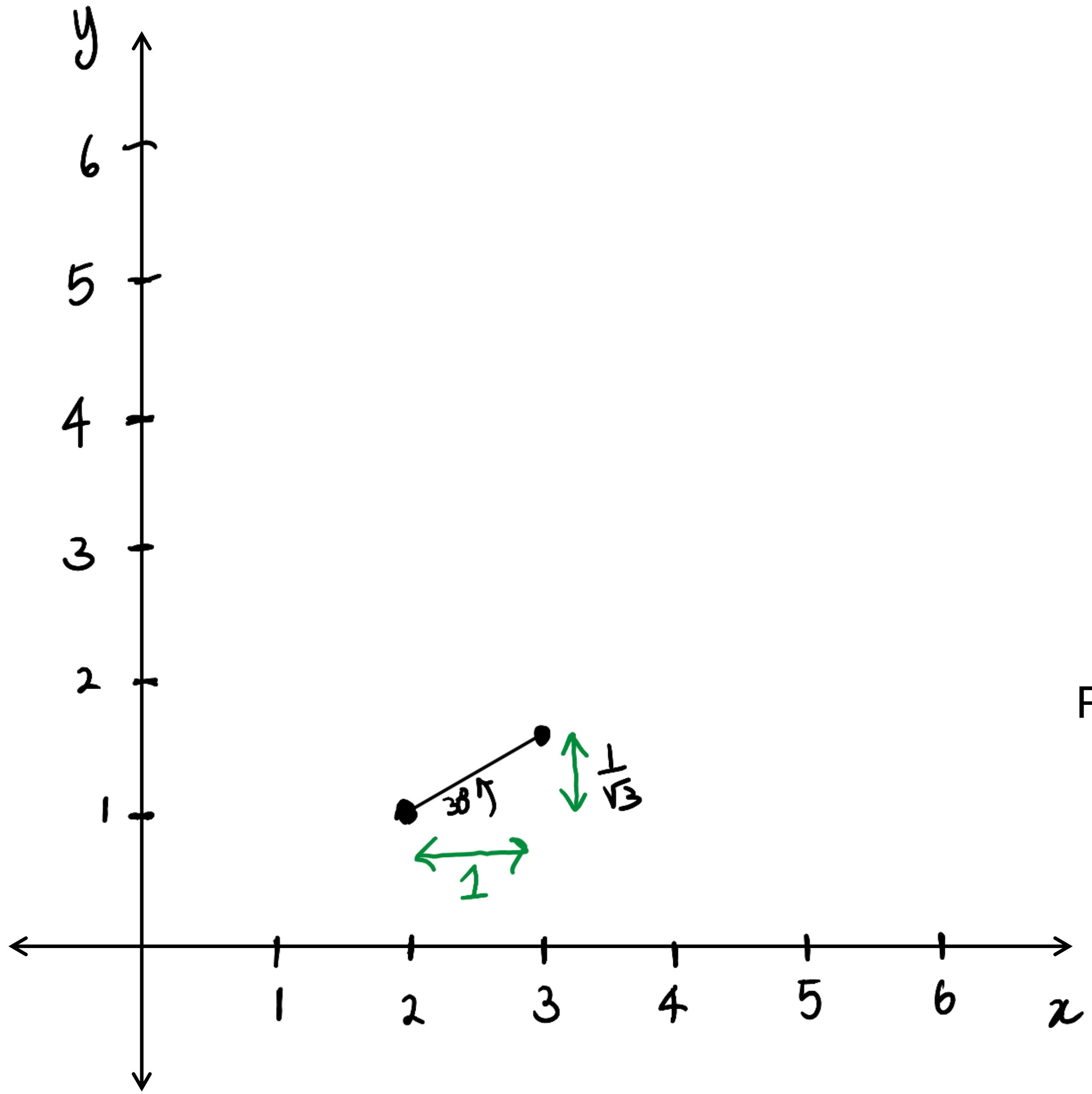
$\tan(\theta) = \frac{o}{a}$

points $(2,1)$ and $\left(3, 1+\frac{1}{\sqrt{3}}\right)$

$$1+\frac{1}{\sqrt{3}}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}}$$

points $(2,1)$ and $\left(3, \frac{\sqrt{3}+1}{\sqrt{3}}\right)$



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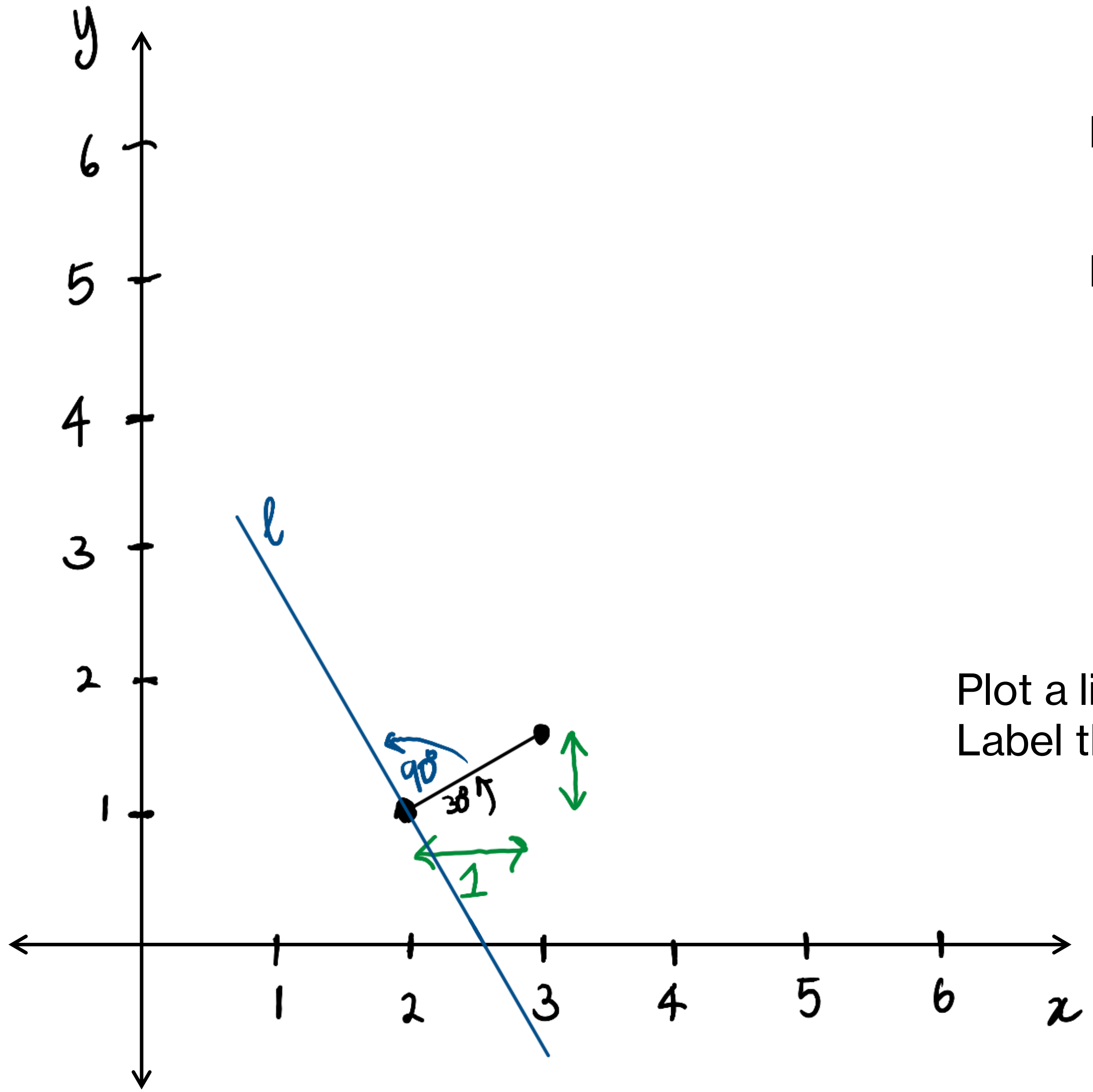
Plot a line perpendicular to the line segment we have draw



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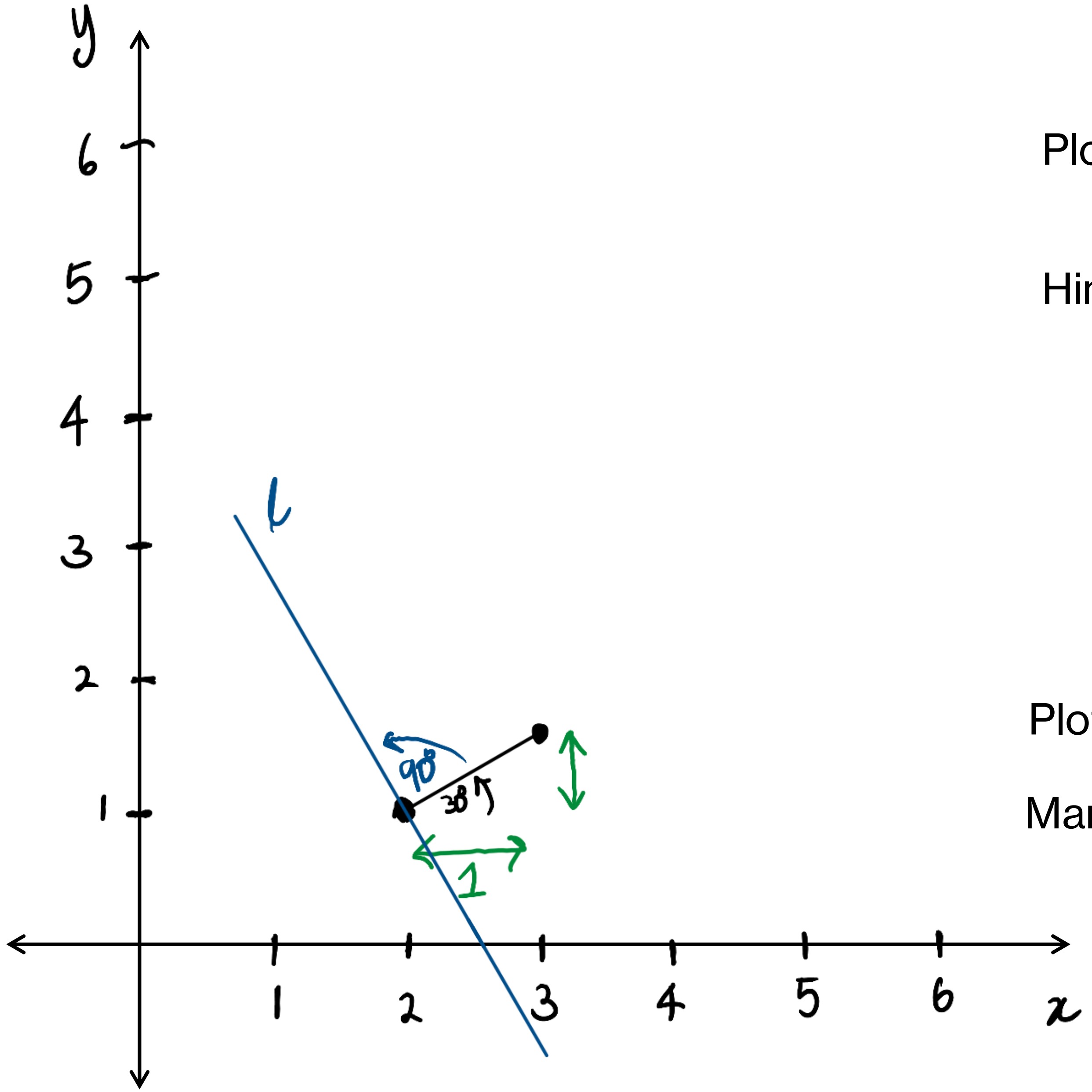
points $(2,1)$ and $\left(3, \frac{\sqrt{3}+1}{\sqrt{3}}\right)$

Plot a line perpendicular to the line segment we have drawn.
Label this line l



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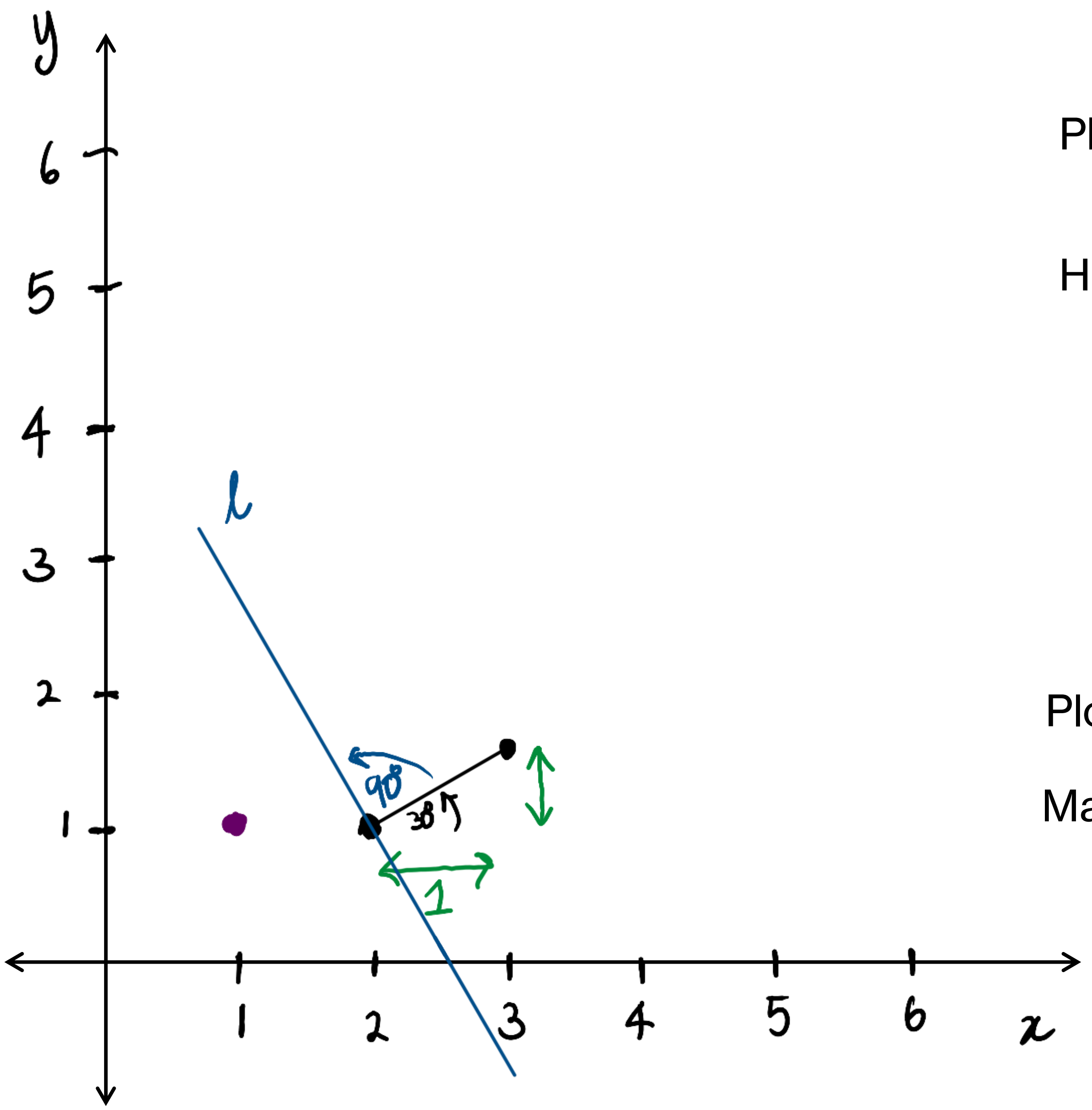
points $(2,1)$ and $\left(3, \frac{\sqrt{3}+1}{\sqrt{3}}\right)$

Plot a line perpendicular to the line segment we have draw

Mark the point (1, 1)



Plot two points so the angle of the slope is 30°

Hint: tan (30 °) = $\frac{1}{\sqrt{3}}$

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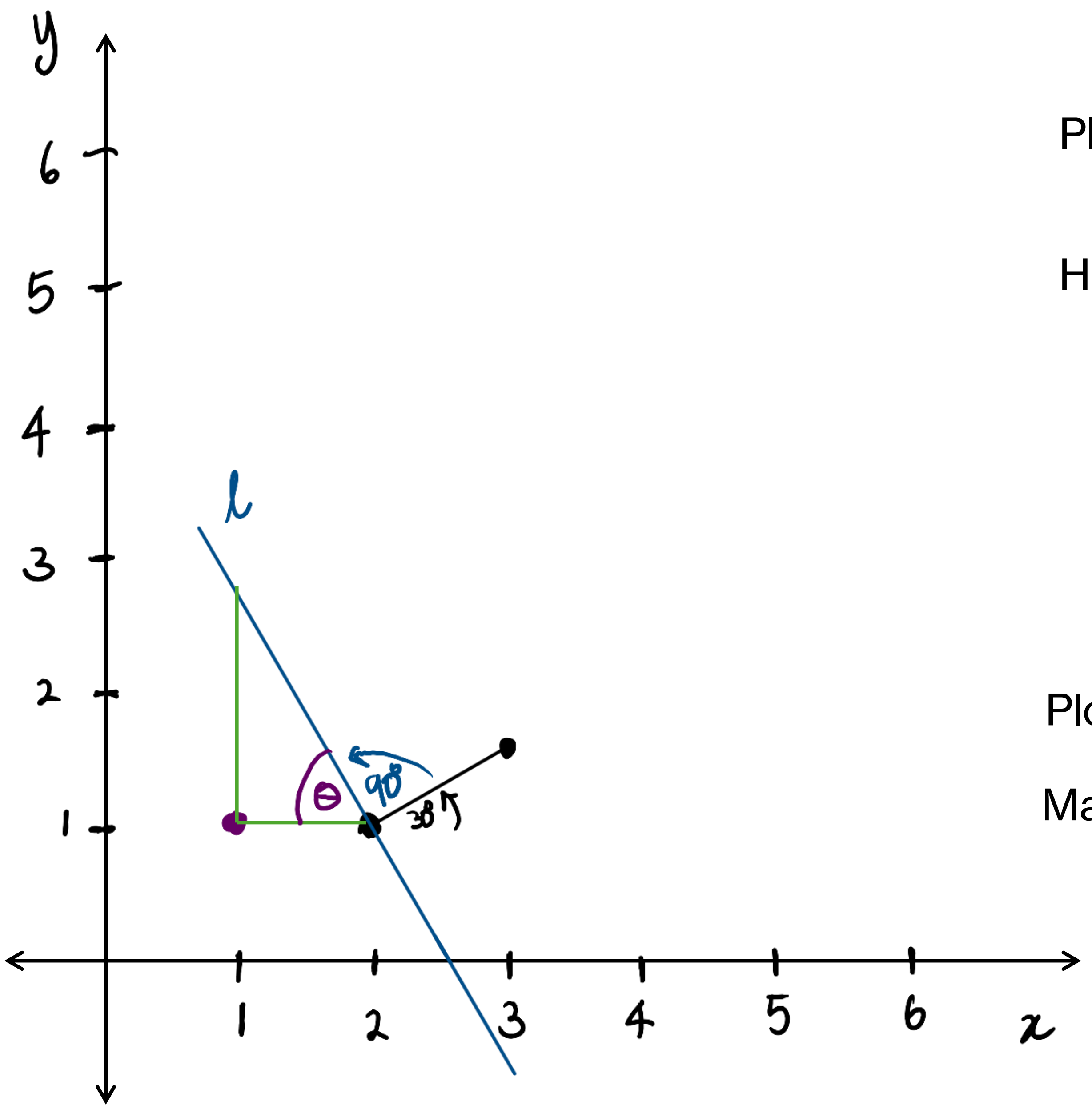
points $(2,1)$ and $\left(3, \frac{\sqrt{3}+1}{\sqrt{3}}\right)$

Plot a line perpendicular to the line segment we have draw

Mark the point (1, 1)

Form a right-angled triangle, using the points (1,1) and (2, 1) and the line l



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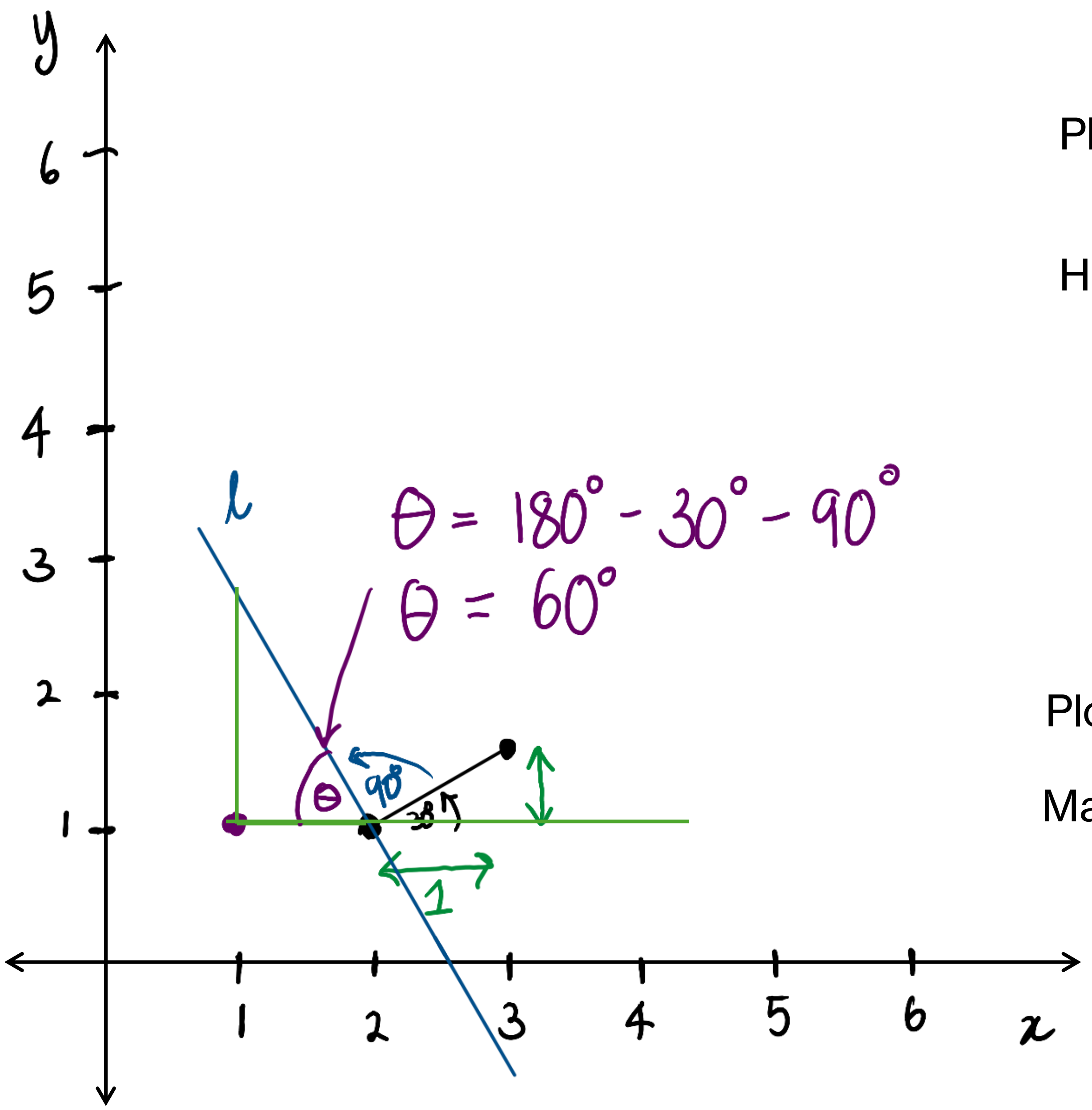
Mark the point (1, 1)

Form a right-angled triangle, using the points (1,1) and (2, 1) and the line l

Calculate the size of the angle, θ



Plot two points so the angle of the slope is 30°

Hint: tan (30 °) = $\frac{1}{\sqrt{3}}$

$\tan(\theta) = \frac{o}{a}$

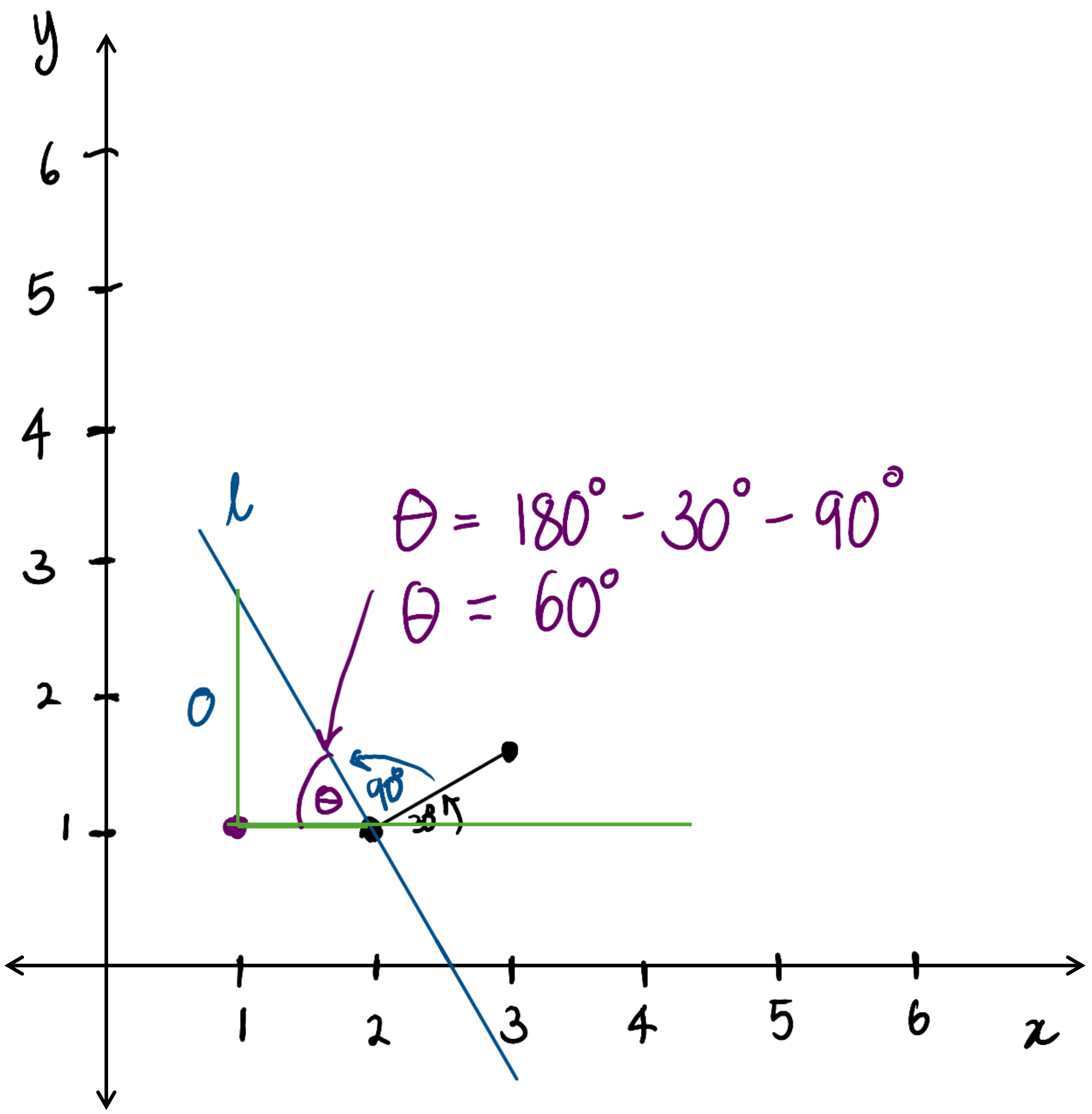
points $(2,1)$ and $\left(3, \frac{\sqrt{3}+1}{\sqrt{3}}\right)$

Plot a line perpendicular to the line segment we have draw

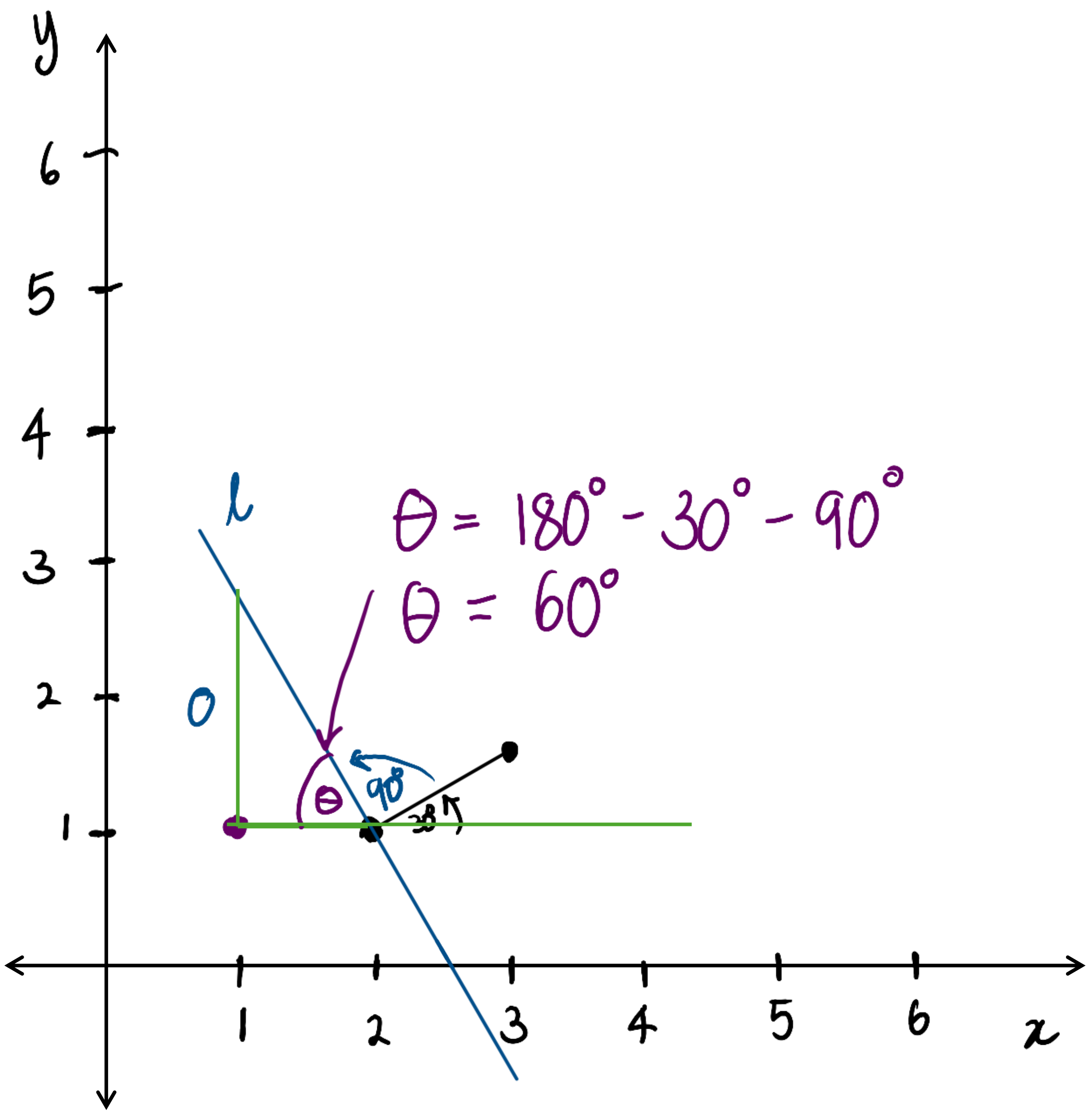
Mark the point (1, 1)

Form a right-angled triangle, using the points (1,1) and (2, 1) and the line l

Calculate the size of the angle, θ

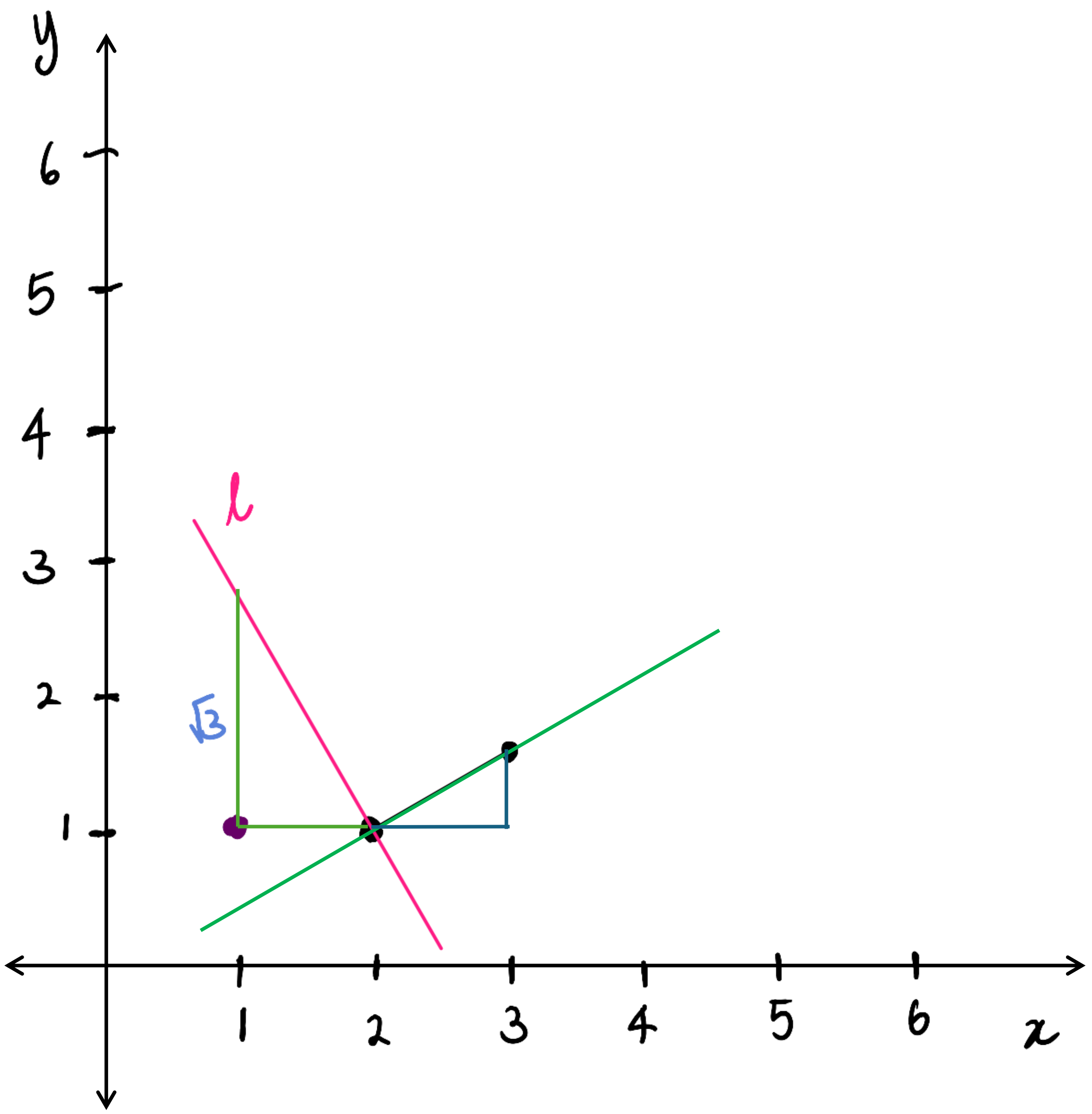


Calculate the length of the side, o



Calculate the length of the side, o

Hint: $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$



Calculate the length of the side, o

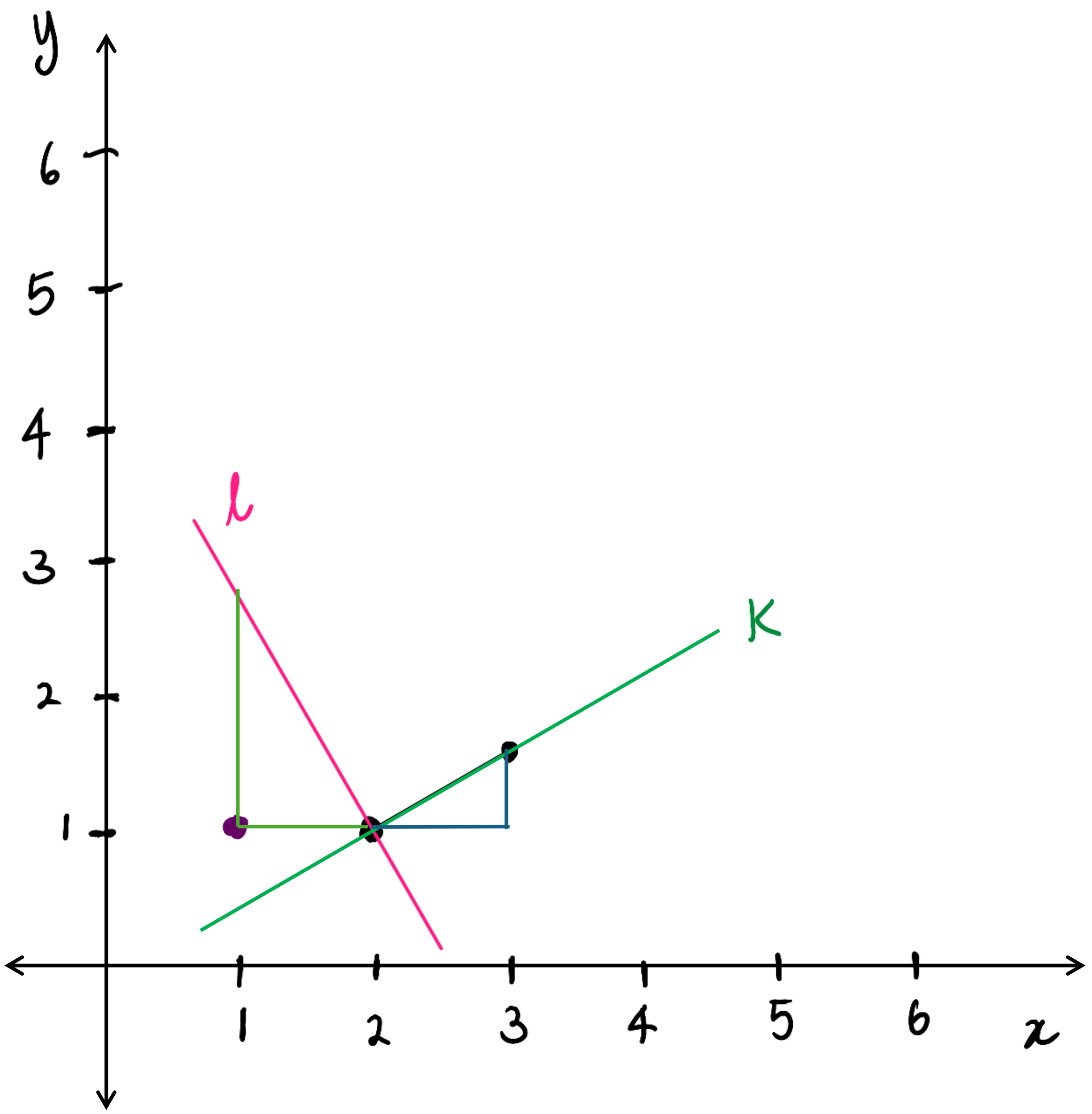
Hint: $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$

$\tan(60^\circ) = \frac{o}{a}$

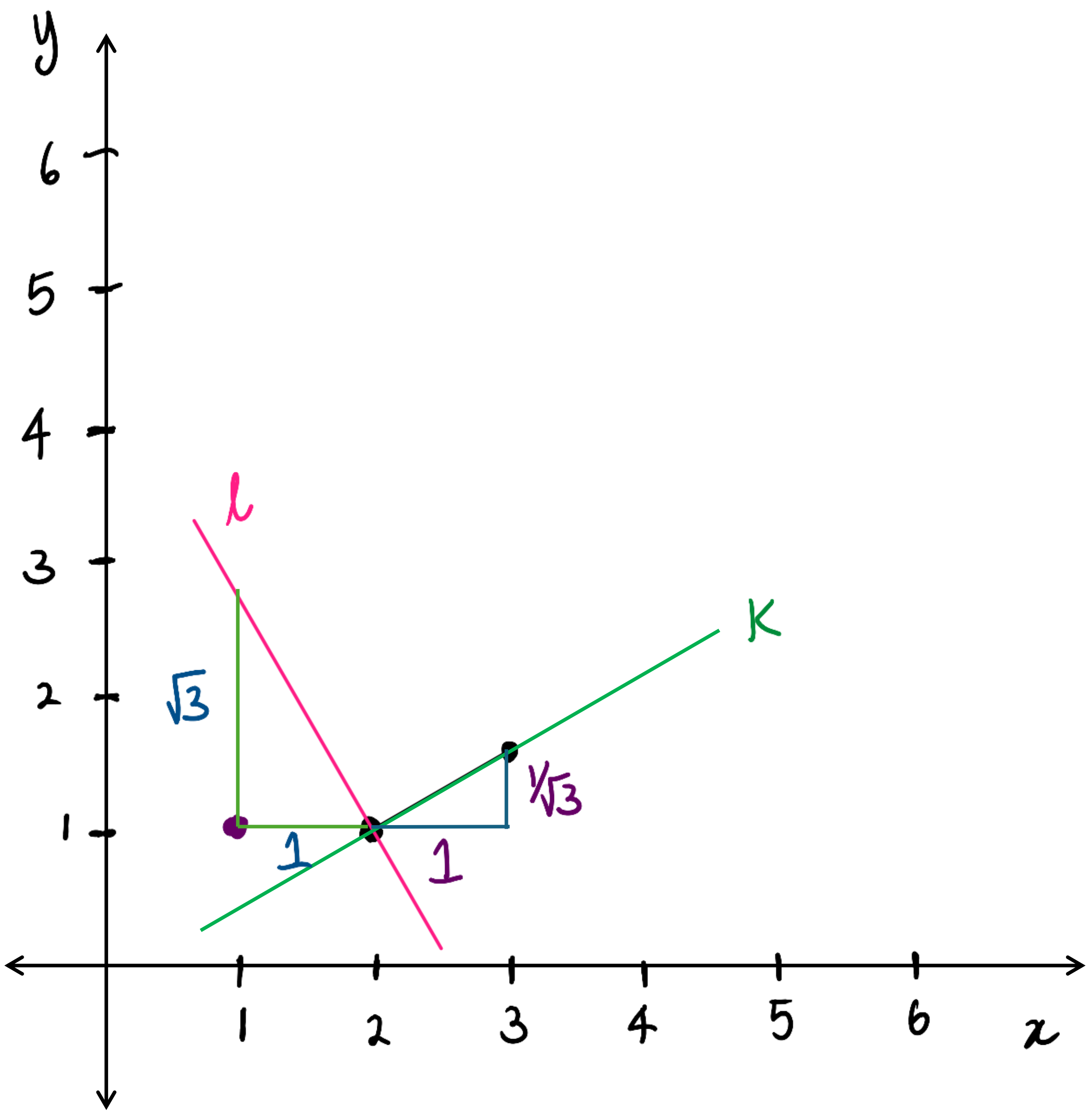
$\tan(60^\circ) = \sqrt{3}$

$a = 1$

$\therefore \; o = \sqrt{3}$



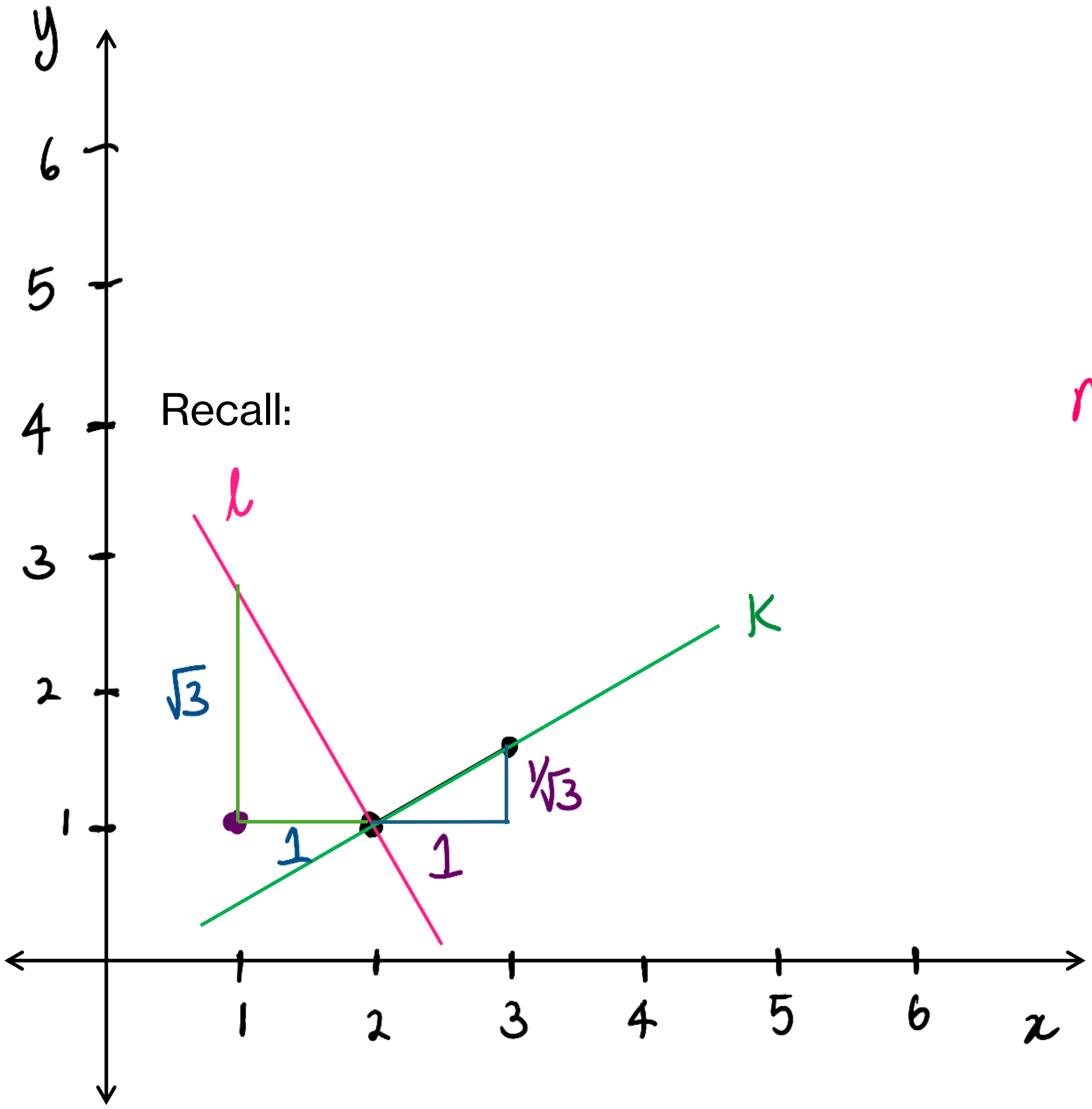
Find the slopes of the lines, l and k



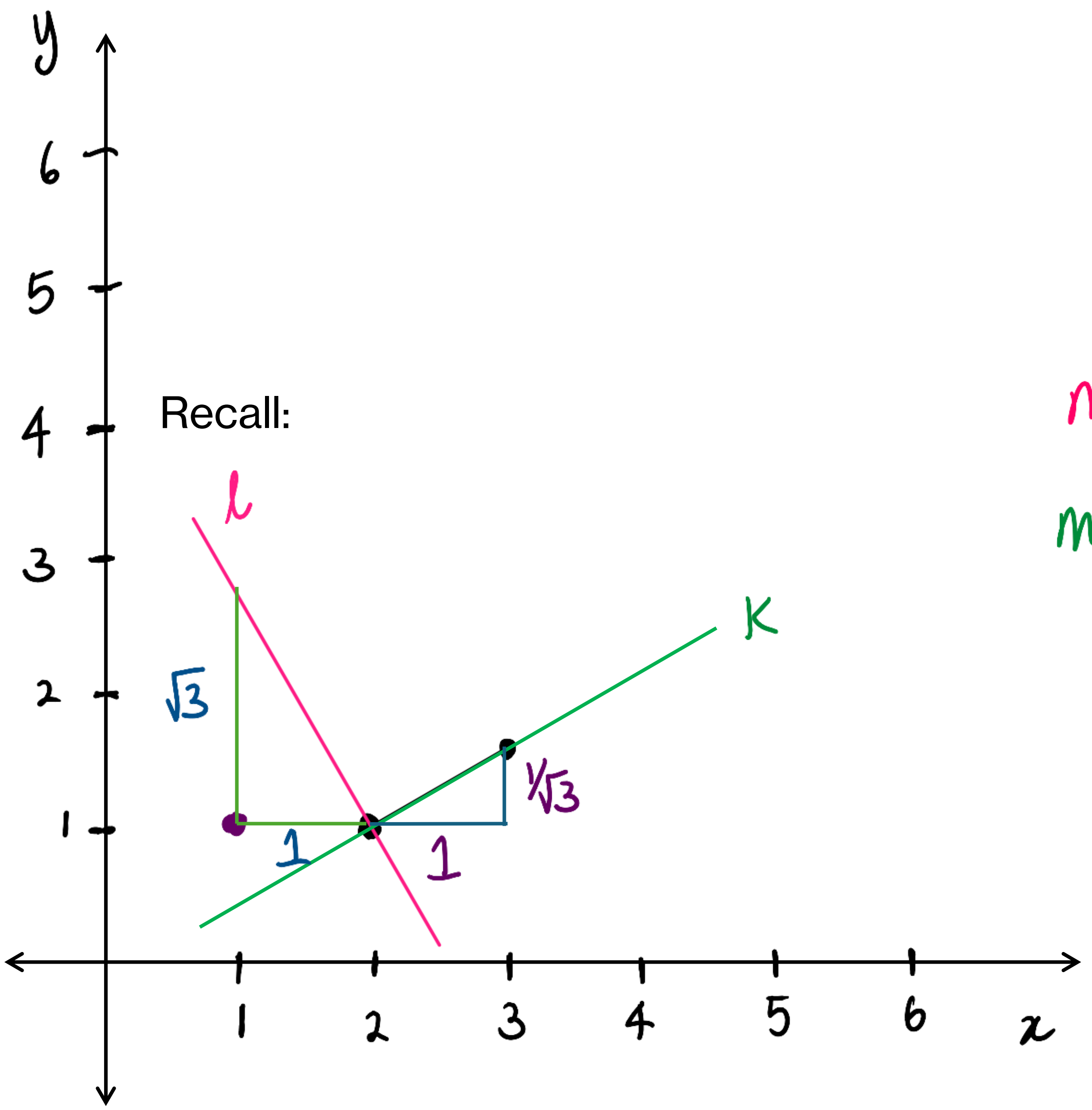
Find the slopes of the lines, l and k

Find the slopes of the lines, l and k

Recall:



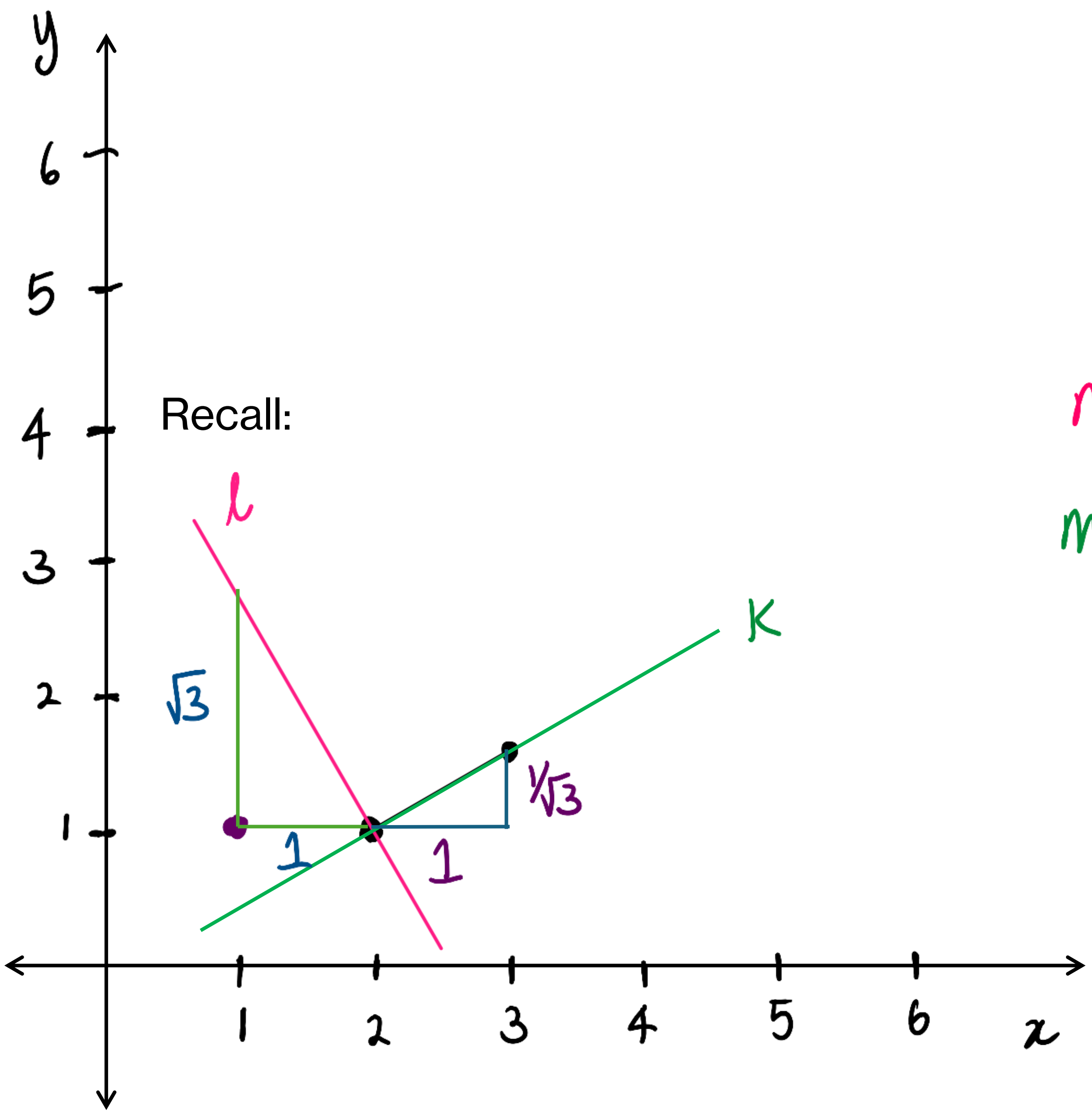
$$m_l = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$



Find the slopes of the lines, l and k

$$m_l = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$

$$m_k = \frac{\frac{1}{\sqrt{3}}}{1} = \frac{1}{\sqrt{3}}$$



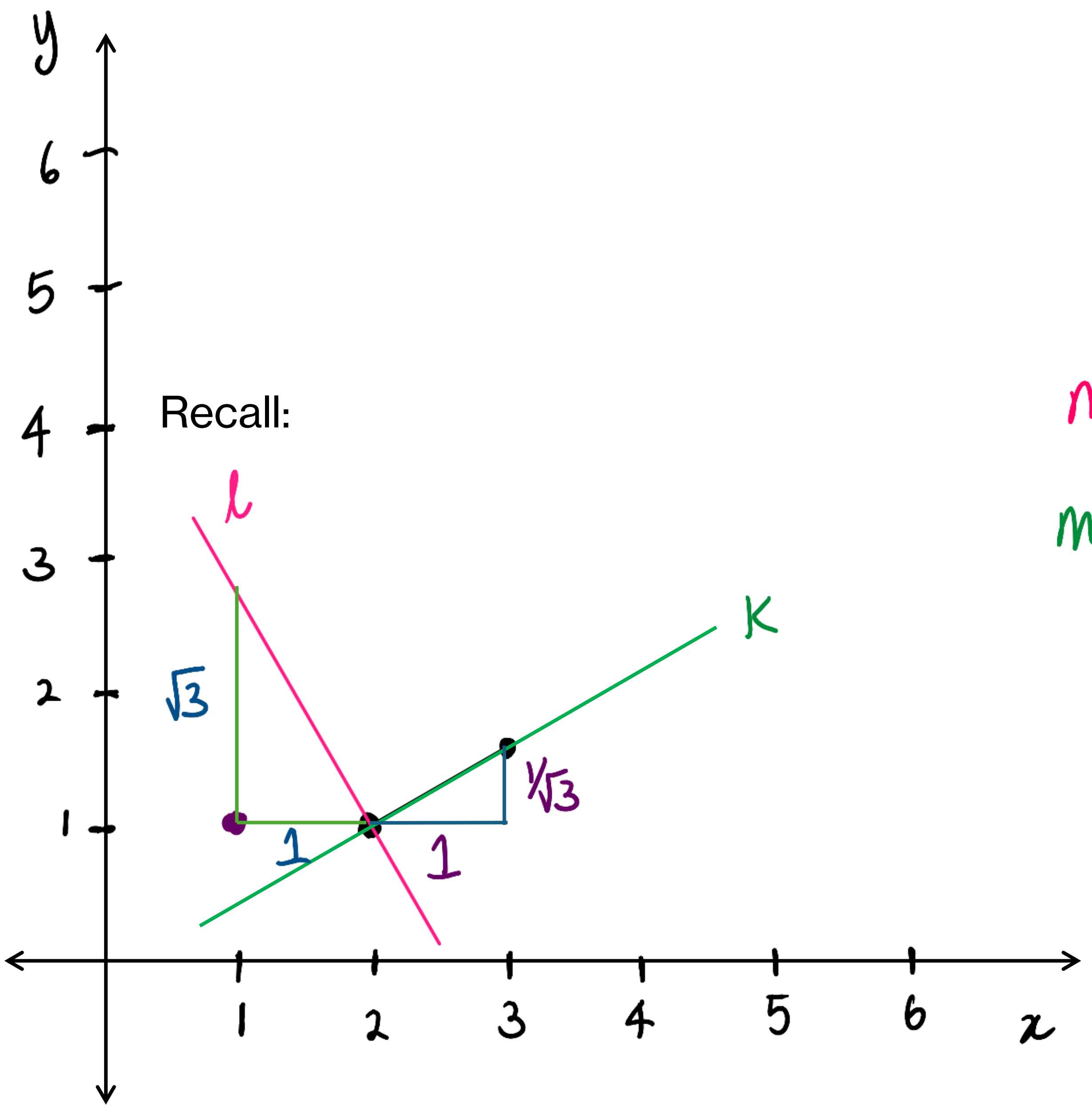
Find the slopes of the lines, l and k

$$m_l = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$

$$m_k = \frac{\frac{1}{\sqrt{3}}}{1} = \frac{1}{\sqrt{3}}$$

Find the product of the slopes of the lines, l and k

Recall:



Find the slopes of the lines, l and k

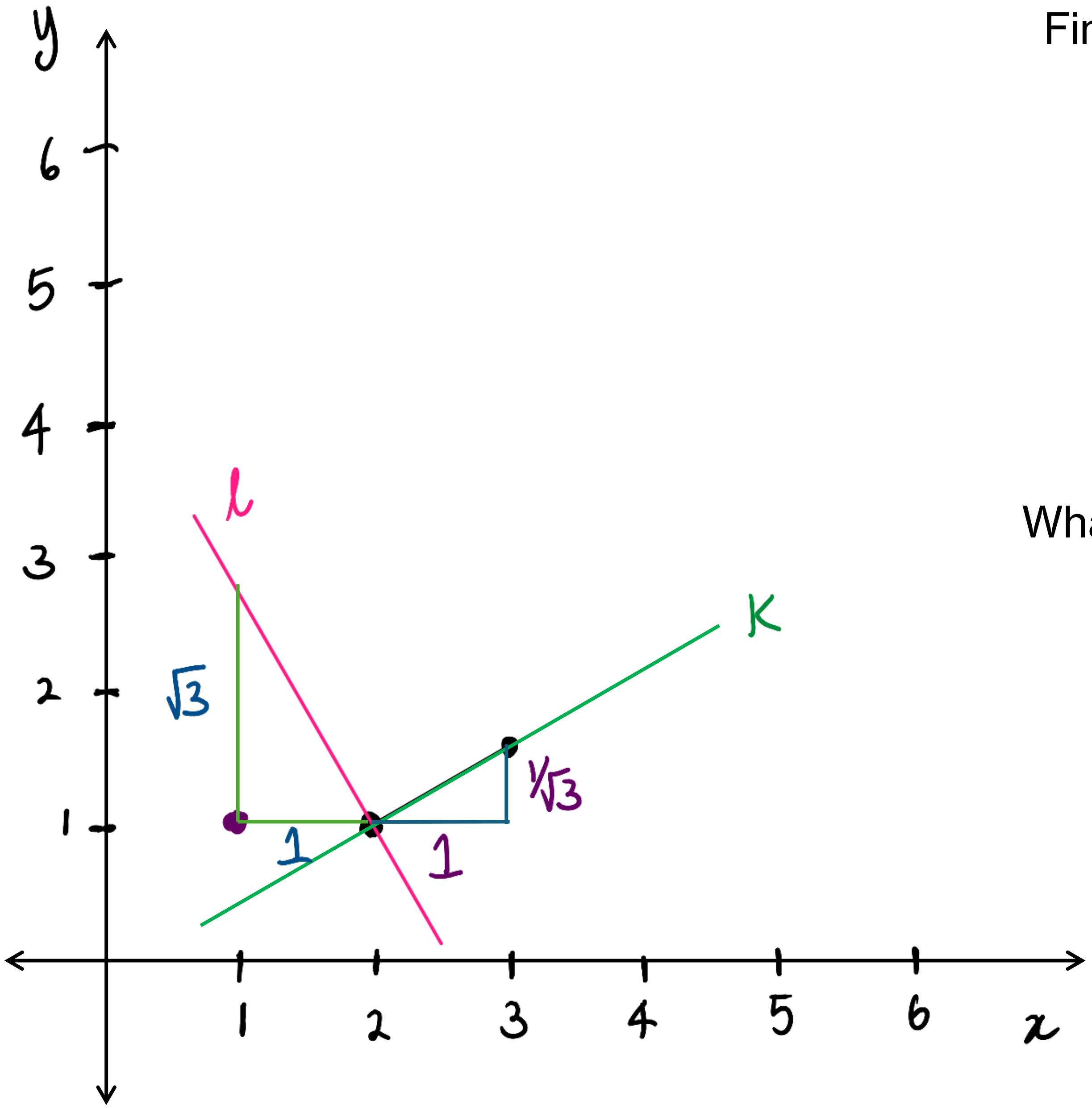
$$m_l = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$

$$m_k = \frac{\frac{1}{\sqrt{3}}}{1} = \frac{1}{\sqrt{3}}$$

Find the product of the slopes of the lines, l and k

$$\frac{1}{\sqrt{3}} \cdot -\sqrt{3} = \frac{-\sqrt{3}}{\sqrt{3}}$$

$$= -1$$

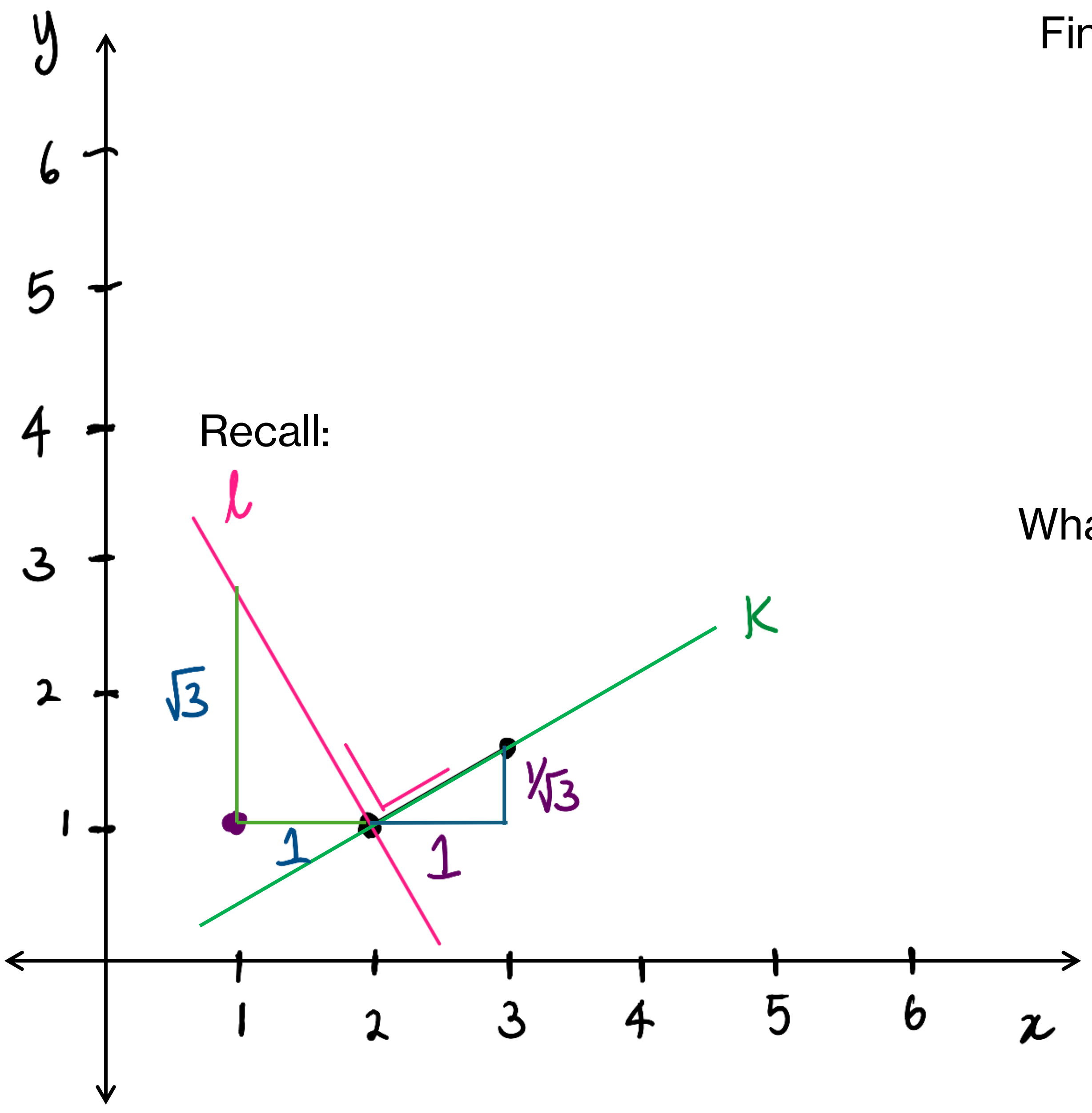


Find the product of the slopes of the lines, l and k

$$\frac{1}{\sqrt{3}} \cdot -\sqrt{3} = \frac{-\sqrt{3}}{\sqrt{3}} = -1$$

What is the relationship between the lines, l and k

Recall:

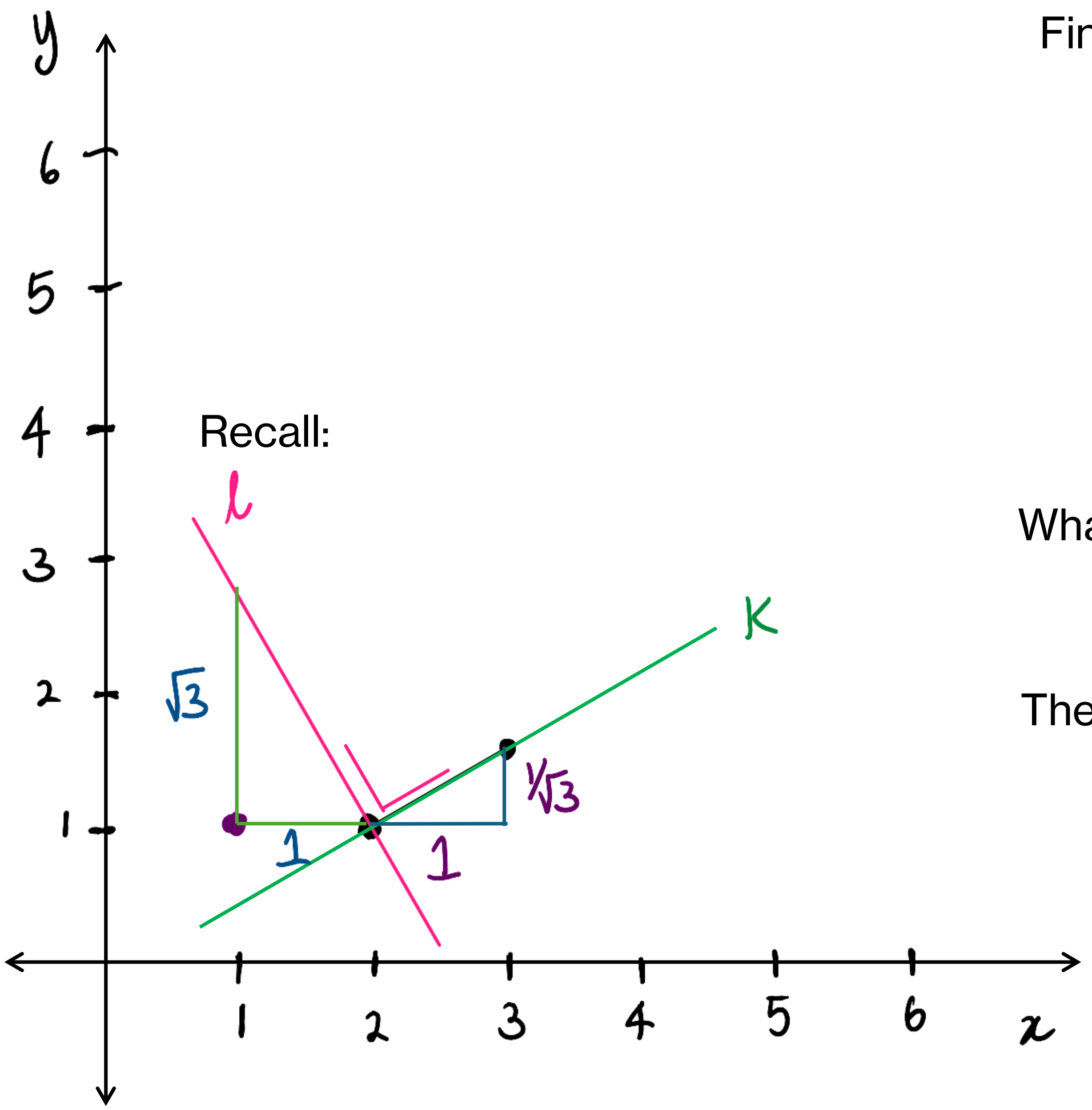


Find the product of the slopes of the lines, l and k

$$\frac{1}{\sqrt{3}} \cdot -\sqrt{3} = \frac{-\sqrt{3}}{\sqrt{3}} = -1$$

What is the relationship between the lines, l and k

Recall:



Find the product of the slopes of the lines, l and k

$$\frac{1}{\sqrt{3}} \cdot -\sqrt{3} = \frac{-\sqrt{3}}{\sqrt{3}} = -1$$

What is the relationship between the lines, l and k

The product of the slopes of perpendicular lines is -1

Building to relational understanding

From proportion to trigonometric functions

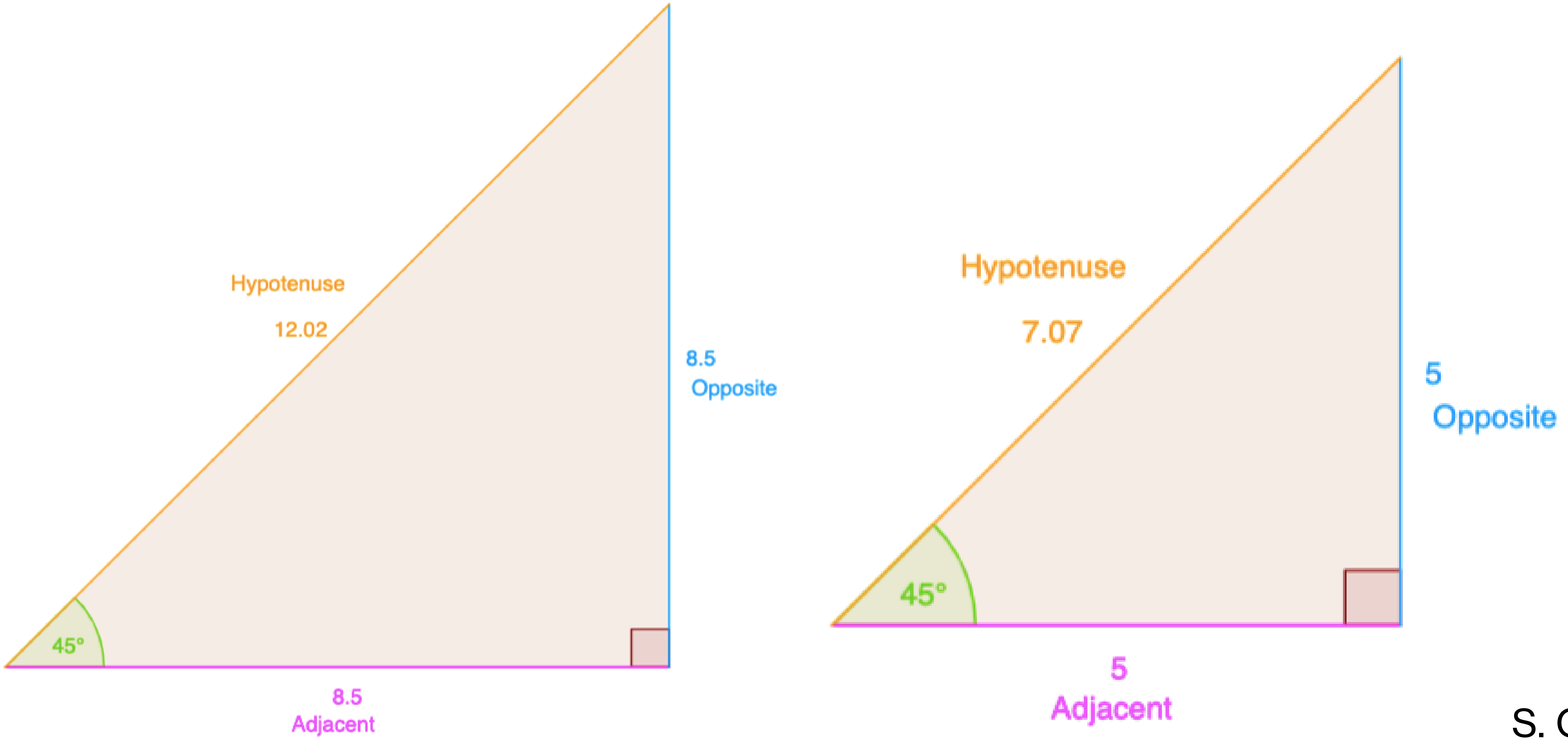
Task

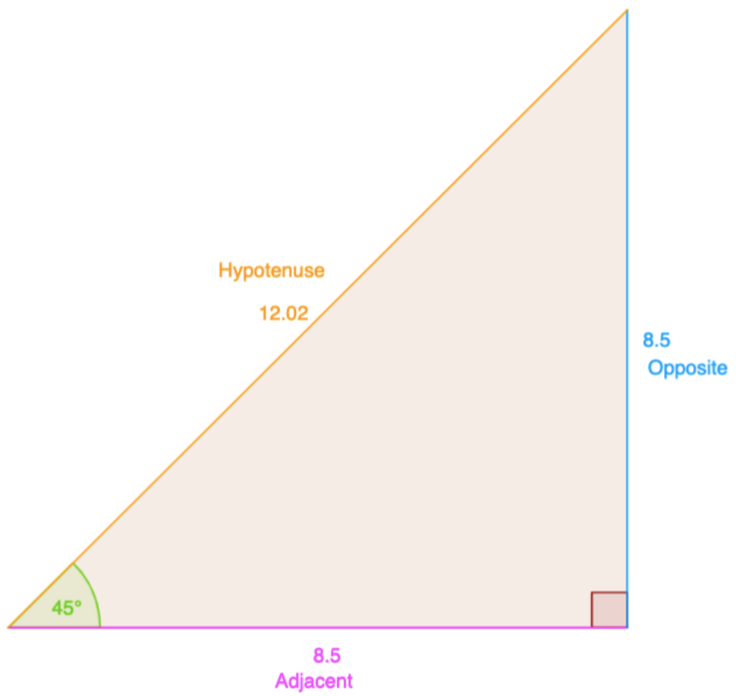
A students asks you, “what is an angle?”

How do you reply?

Studying Triangles

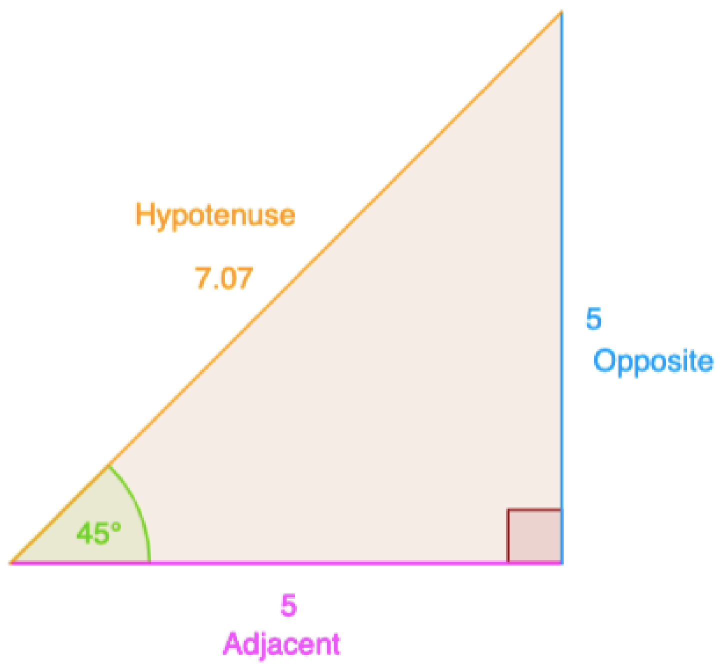
What do you notice about these triangles?





$$\frac{\text{opposite}}{\text{adjacent}} = \qquad \frac{\text{adjacent}}{\text{hypotenuse}} =$$

$$\frac{\text{opposite}}{\text{hypotenuse}} =$$



$$\frac{\text{opposite}}{\text{adjacent}} = \qquad \frac{\text{adjacent}}{\text{hypotenuse}} =$$

$$\frac{\text{opposite}}{\text{hypotenuse}} =$$

Ratio and Proportion

- A **ratio** is a relationship between constituent parts.
- Proportion refers to the relationship or ratio between different elements or parts of a whole.
- It involves comparing the size, quantity, or magnitude of one component to another or to the entirety.
- Proportions are often expressed in terms of percentages, fractions, or ratios.

What does the following mean: $\sin\frac{\pi}{4}$

$$h(t) = 72 - 60\cos\left(\frac{\pi}{3}t\right)$$

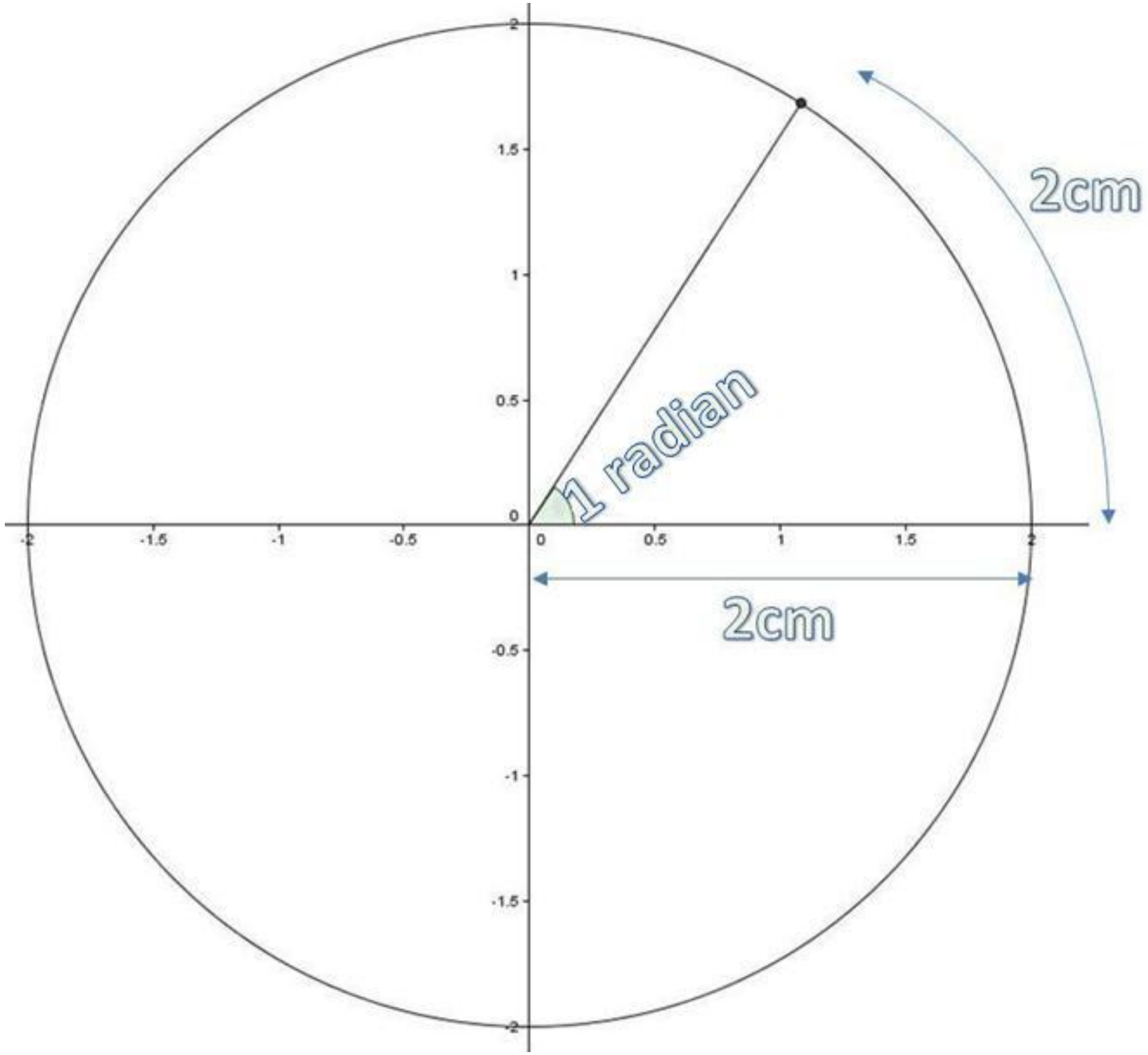
where h is in metres, $t \in \mathbb{R}$, and $\frac{\pi}{3}t$ is in radians.

What is a radian?

Why is the function input in radians?

Standard units of measurement

Quantity	The International System of Units (SI) Unit
Unit of mass	kilogram
Unit of length	metre
Unit of time	second
Unit of electric current	ampere
Unit of temperature	kelvin
Unit of angular measurement	**radian**



Examine the image on the left.

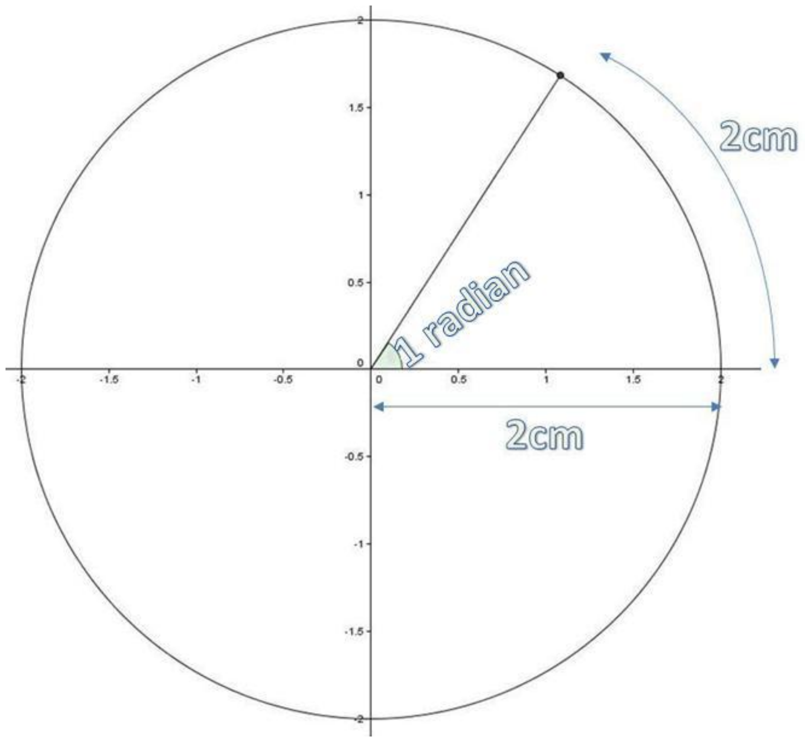
What do you notice about radians?

Radians

- A radian is a unit of angular measure based on the radius of a circle.

- One radian is the angle made at the center of a circle by an arc whose length is equal to the radius of the circle.

- To find the radian measure of an angle, therefore, we use the formula $\frac{arc\ length}{radius}$



$$1\ radian = \frac{2\ cm}{2\ cm}$$

What is a radian?

If we had a unit circle what would be the angle at O, the centre of the circle?

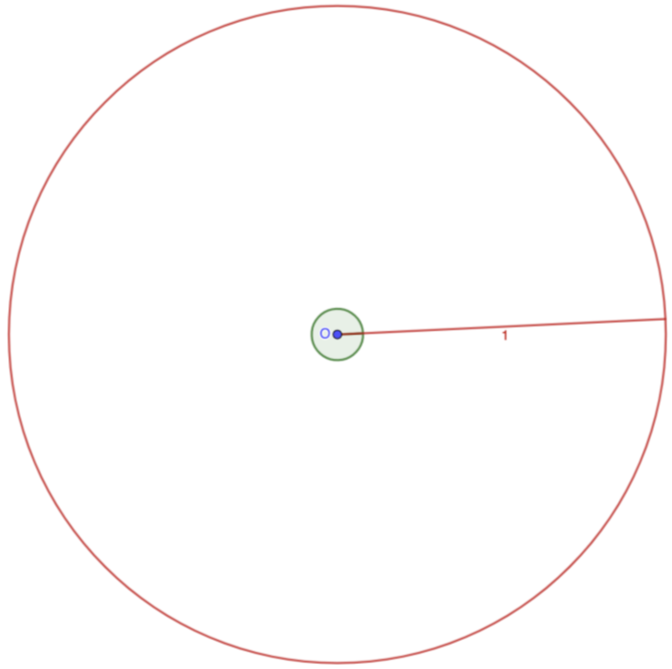
360°

What is another term for the arc enclosing *O*?

Circumference

What is the length of this arc, given it is a **unit** circle?

$2\pi(1) = 2\pi$



Summary: Radians in a Circle

We know the circumference of a circle is given by the formula

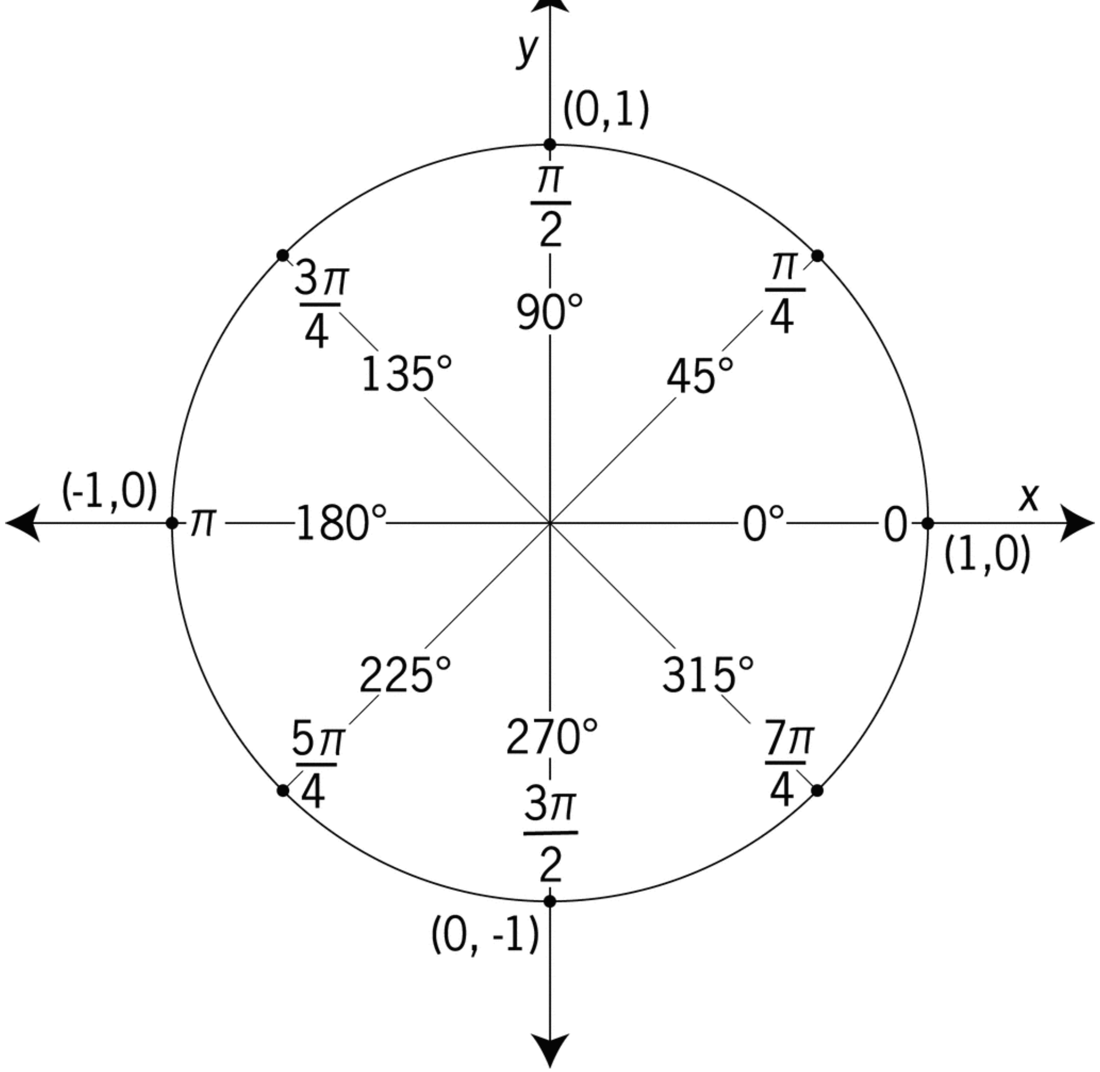
$$C = 2\pi r$$

Angle in radian measure $= \frac{arc\ length}{radius}$

Angle in a circle $= \frac{2\pi r}{r} = 2\pi$

Therefore, there are 2π radians in a circle.

Any angle can be measured in radians or degrees



Converting Degrees to Radians

We can convert degrees to radians.

Convert $135°$ to radians.

$$180° = \pi \text{ rad}$$

$$1° = \frac{\pi}{180} \text{ rad}$$

$$135° = \frac{135\pi}{180} \text{ rad}$$

$$135° = \frac{3\pi}{4} \text{ rad}$$

Task

Convert $210°$ to radians.

$$180° = \pi \text{ rad}$$

$$1° = \frac{\pi}{180} \text{ rad}$$

$$210° = \frac{210\pi}{180} \text{ rad}$$

$$210° = \frac{7\pi}{6} \text{ rad}$$

In general what method would we use if we wanted to convert $x°$ to radians?

$$x \times \frac{\pi}{180}$$

Converting Radians to Degrees

We can convert radians to degrees also.

Convert $\frac{8\pi}{6}\,\text{rad}$ to degrees.

$$\pi\ \text{rad} = 180°$$

$$\frac{8\pi}{6}\text{rad} = \frac{8(180)}{6}$$

$$\frac{8\pi}{6}\text{rad} = \frac{1440}{6}$$

$$\frac{8\pi}{6}\text{rad} = 240°$$

Task

Convert $\frac{7\pi}{5}$ radians to degrees

$$\pi \text{ rad} = 180°$$

$$\frac{7\pi}{5}\text{rad} = \frac{7(180)}{5}$$

$$\frac{7\pi}{5}\text{rad} = \frac{1260}{6}$$

$$\frac{7\pi}{5}\text{rad} = 210°$$

In general what method would we use if we wanted to convert r radians to degrees?

$$r \times \frac{180}{\pi}$$

Examine the following applet:

What do you notice?

- https://www.geogebra.org/m/UjjwuM8p

Obscure:

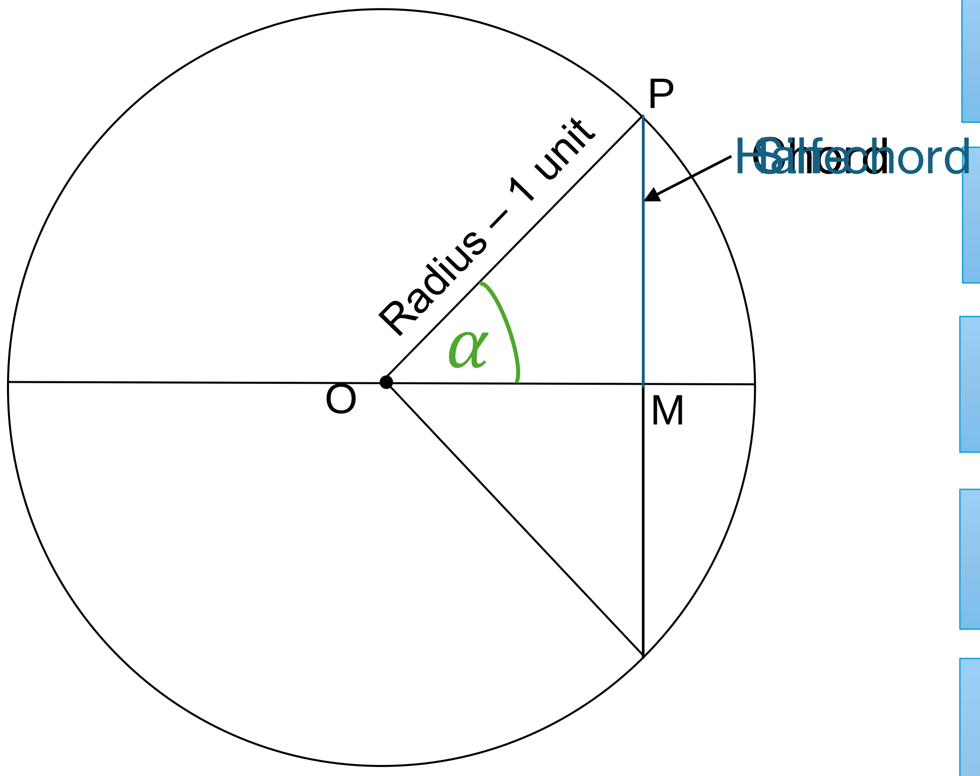
Adj. not clearly expressed or easily understood

Verb. Keep from being seen

To **reveal** the obscure:

Trigonometry and the Unit Circle

The origins of trigonometry are closely tied up with problems involving circles.



By drawing a chord and second radius, we form a triangle within the unit circle.

By drawing the diameter, we can form a right angled triangle, OPM.

The diameter is a perpendicular bisector of the chord, forming a **half-chord.**

$$\frac{\text{opposite}}{\text{hypotenuse}} = \frac{PM}{OP} = \frac{\text{Half−chord}}{1}$$

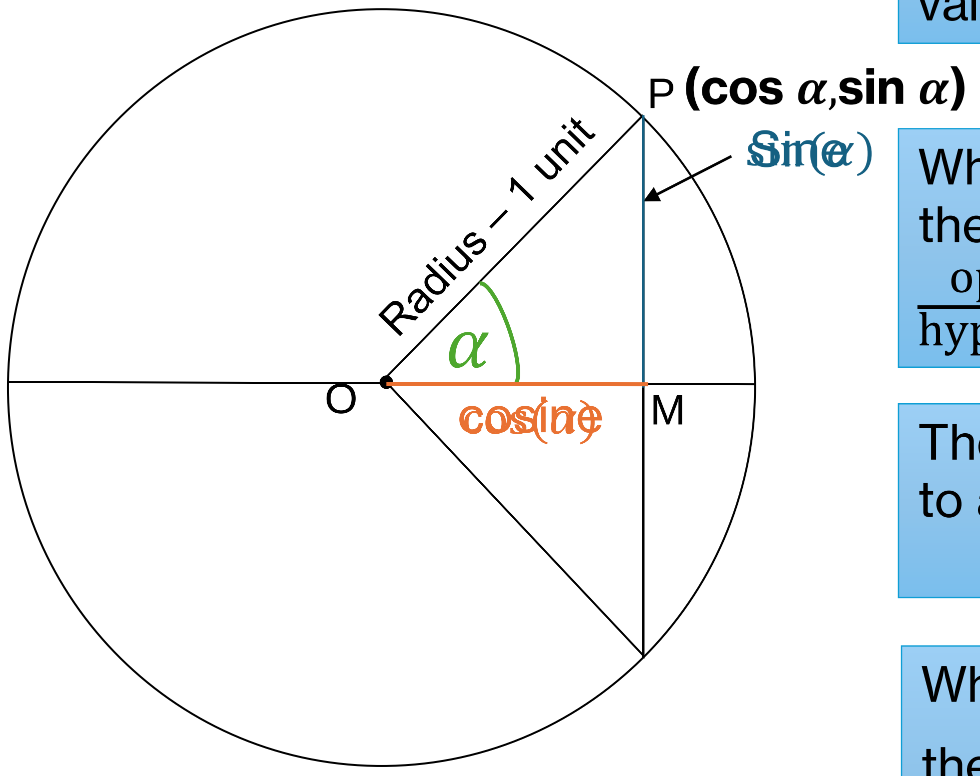
Label $|\angle POM| = \alpha$

For the angle α, the ratio of $\frac{\text{opposite}}{\text{hypotenuse}}$ is a constant value.

Sine is the Sanskrit translation of half-chord.

$$\frac{\text{opposite}}{\text{adjacent}} = \frac{PM}{OP} = \frac{\text{Half−chord}}{1}$$

For the angle α, the ratio of $\frac{\text{opposite}}{\text{hypotenuse}}$ is a constant value.



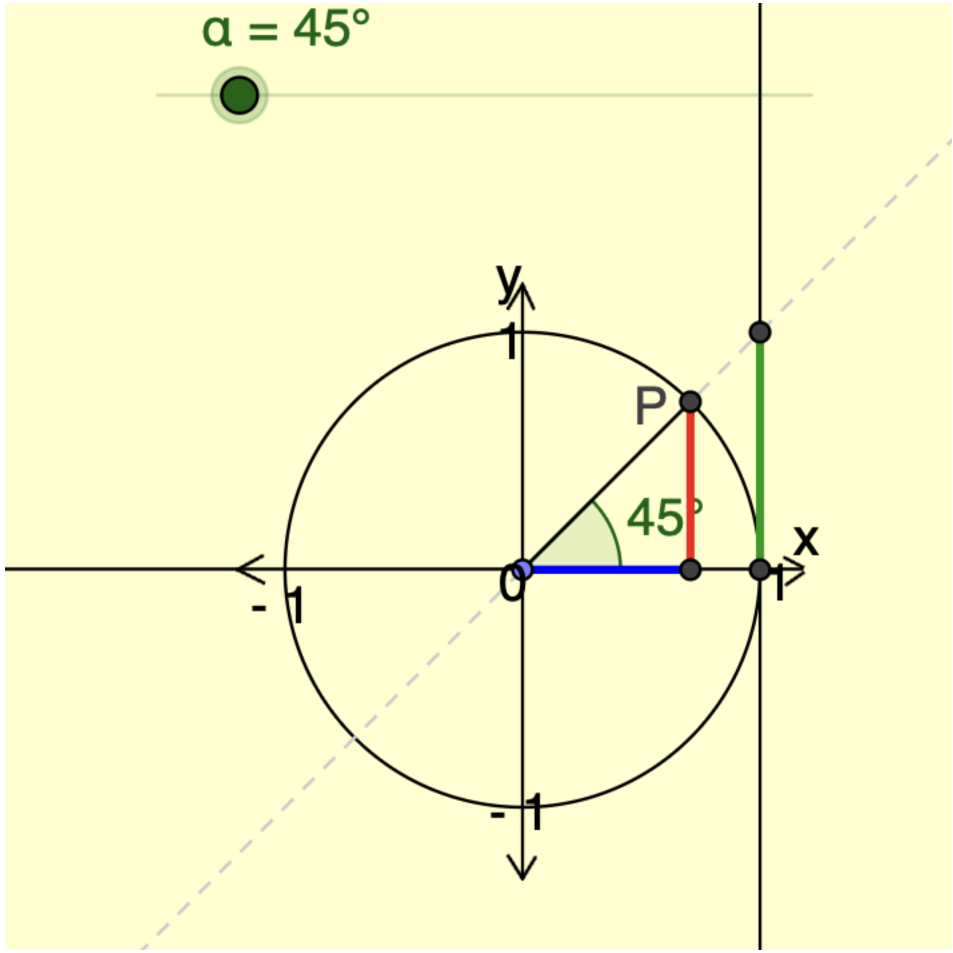
When the value of the hypotenuse is 1, then the height of the half-chord (sine) [opposite] is the value of ratio of $\frac{\text{opposite}}{\text{hypotenuse}}$ for the given angle α.

The triangle is formed by the line OM [adjacent], referred to as the complement of sine, shortened to **cosine**.

When the value of the hypotenuse is 1, then the length of the **cosine** [adjacent] is the value of ratio of $\frac{\text{adjacent}}{\text{hypotenuse}}$ for the given angle α.

The co-ordinates of $P = (\cos(\alpha), \sin(\alpha))$

Angle Measure and the Unit Circle



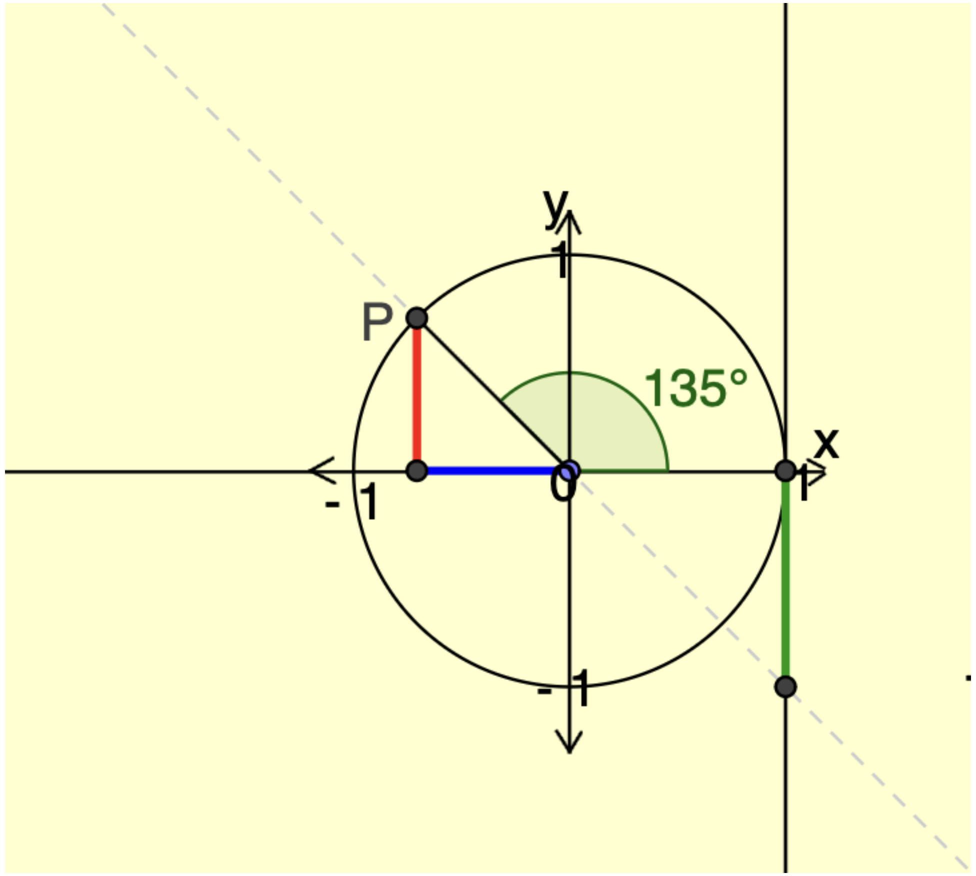
Represent the co-ordinates of the point P using radians.

$$\left(\cos\left(\frac{\pi}{4}\right), \sin\left(\frac{\pi}{4}\right)\right)$$

What can we tell about the values of $\cos\left(\frac{\pi}{4}\right)$ and $\sin\left(\frac{\pi}{4}\right)$?

Why must $\cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right)$

Angle Measure and the Unit Circle



Represent the co-ordinates of the point P using radians.

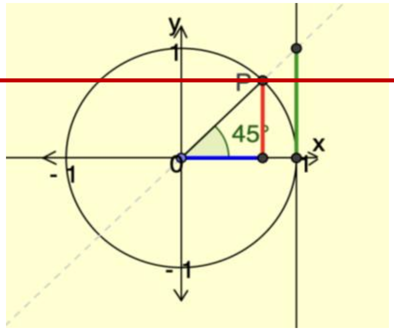
$$\left(\cos\left(\frac{3\pi}{4}\right), \sin\left(\frac{3\pi}{4}\right)\right)$$

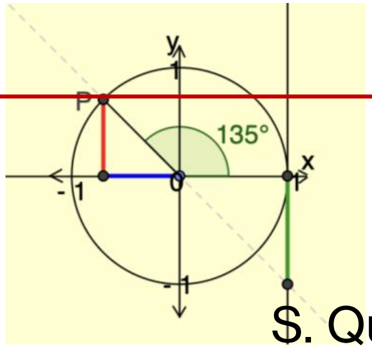
What can we tell about the values of $\cos\left(\frac{3\pi}{4}\right)$ and $\sin\left(\frac{3\pi}{4}\right)$?

$\cos\left(\frac{3\pi}{4}\right)$ is a **negative** value

$\sin\left(\frac{3\pi}{4}\right)$ is a **positive** value

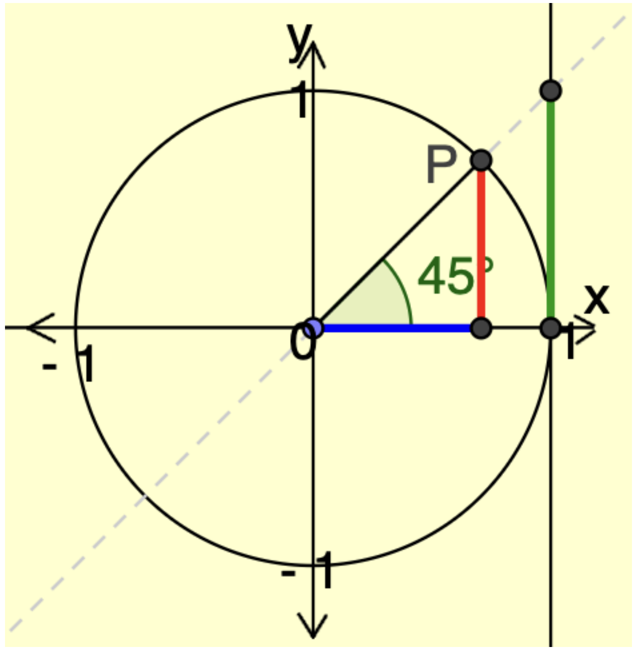
What can we tell about the $\sin\left(\frac{\pi}{4}\right)$ and $\sin\left(\frac{3\pi}{4}\right)$?

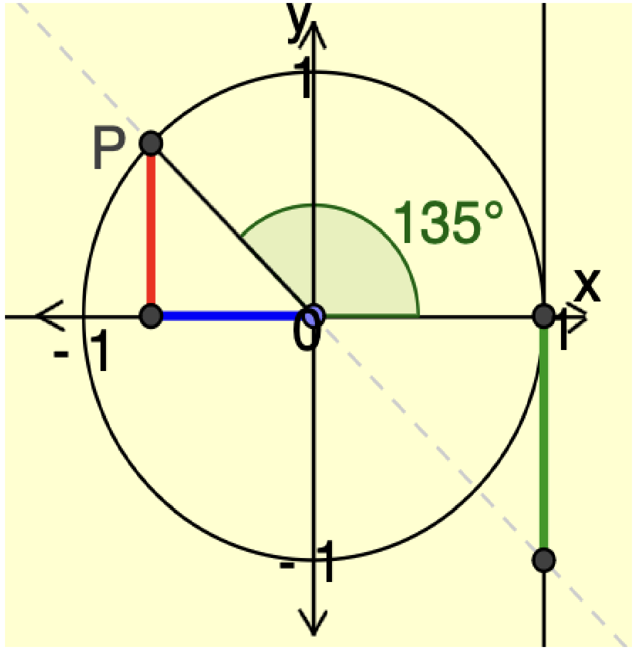




Other Findings

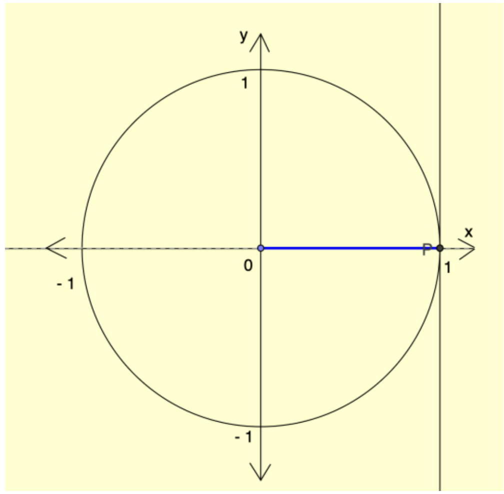
What can we tell about the $\cos\left(\frac{\pi}{4}\right)$ and $\cos\left(\frac{3\pi}{4}\right)$?



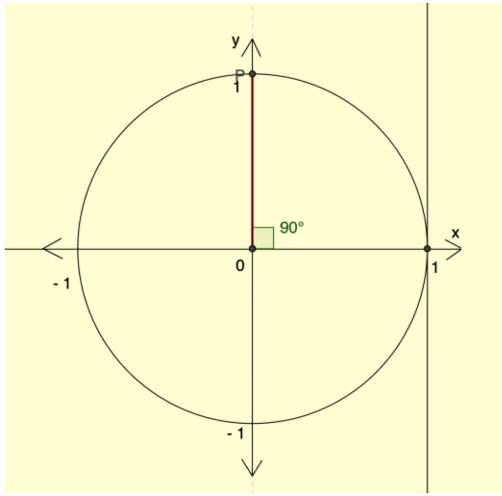


$$-\left(\cos\left(\frac{\pi}{4}\right)\right) = \cos\left(\frac{3\pi}{4}\right)$$

Task

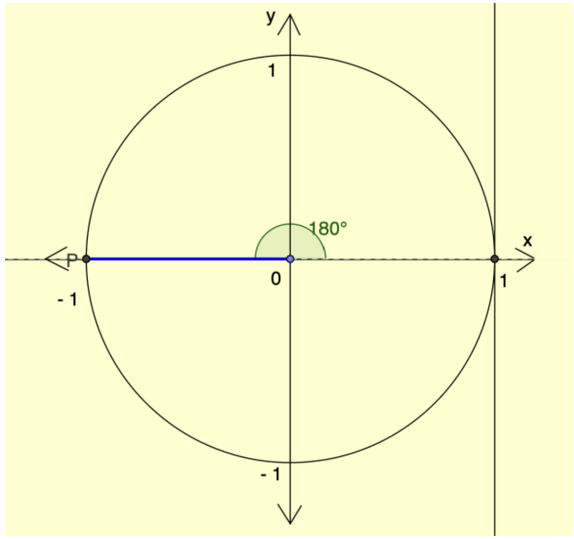


$(\cos(0), \sin(0)) = (1, 0)$

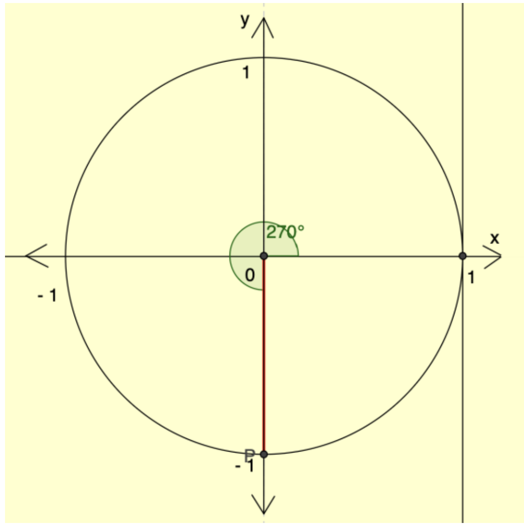


$(\cos(\square), \sin(\square)) = (\square, \square)$

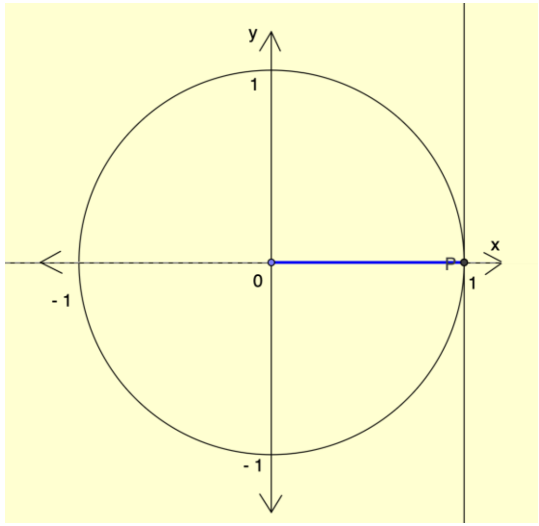
Complete the boxes



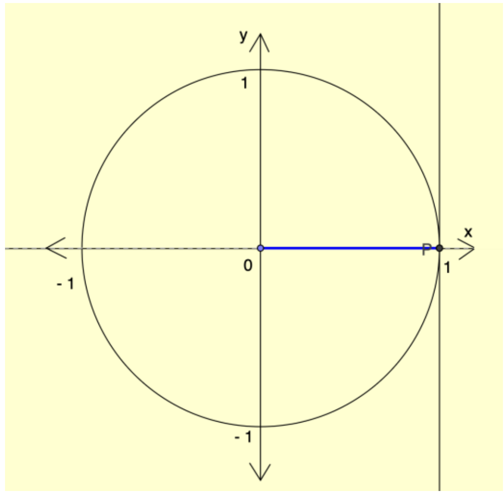
$(\cos(\square), \sin(\square)) = (\square, \square)$



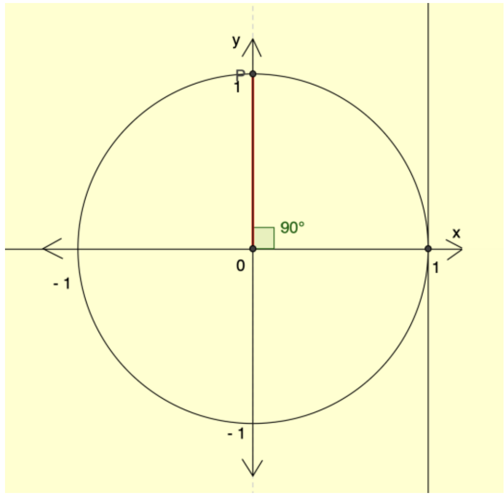
$(\cos(\square), \sin(\square)) = (\square, \square)$



$(\cos(\square), \sin(\square)) = (\square, \square)$



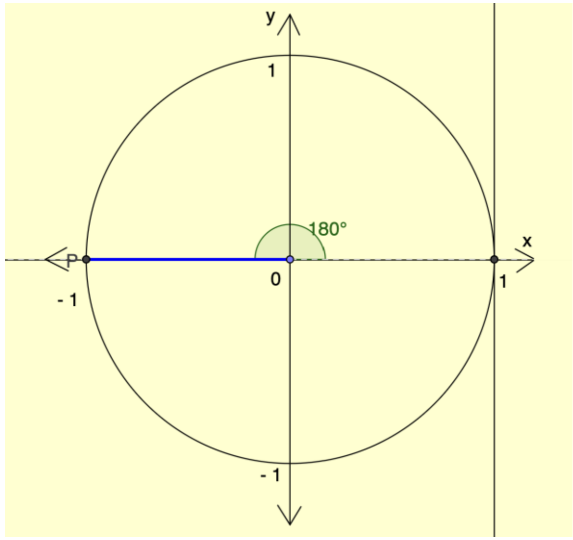
$$(\cos(0), \sin(0)) = (1, 0)$$



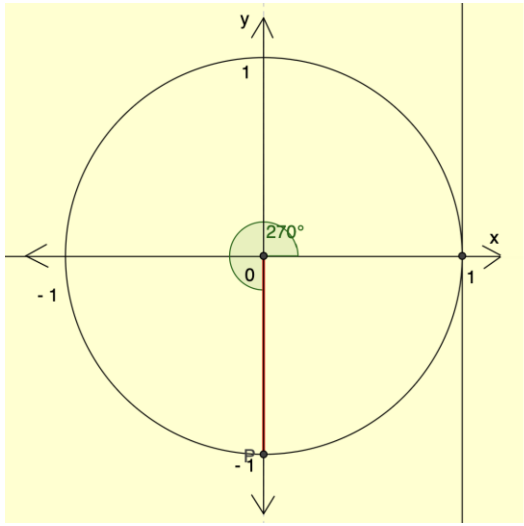
$$\left(\cos\left(\frac{\pi}{2}\right), \sin\left(\frac{\pi}{2}\right)\right) = (0,1)$$

Using this information, draw the graphs of the following functions, for $0 \le \alpha \le 4\pi$:

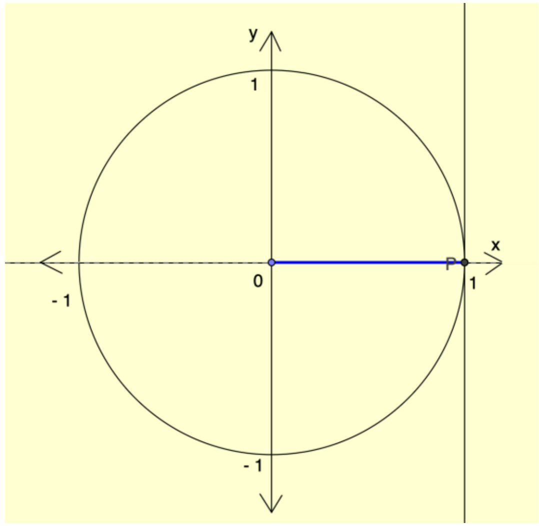
$$f(\alpha): \sin(\alpha)$$
$$g(\alpha): \cos(\alpha)$$



$$(\cos(\pi), \sin(\pi)) = (-1,0)$$



$$\left(\cos\left(\frac{3\pi}{2}\right), \sin\left(\frac{3\pi}{2}\right)\right) = (0,-1)$$



$$(\cos(2\pi), \sin(2\pi)) = (1,0)$$

Trigonometric Functions

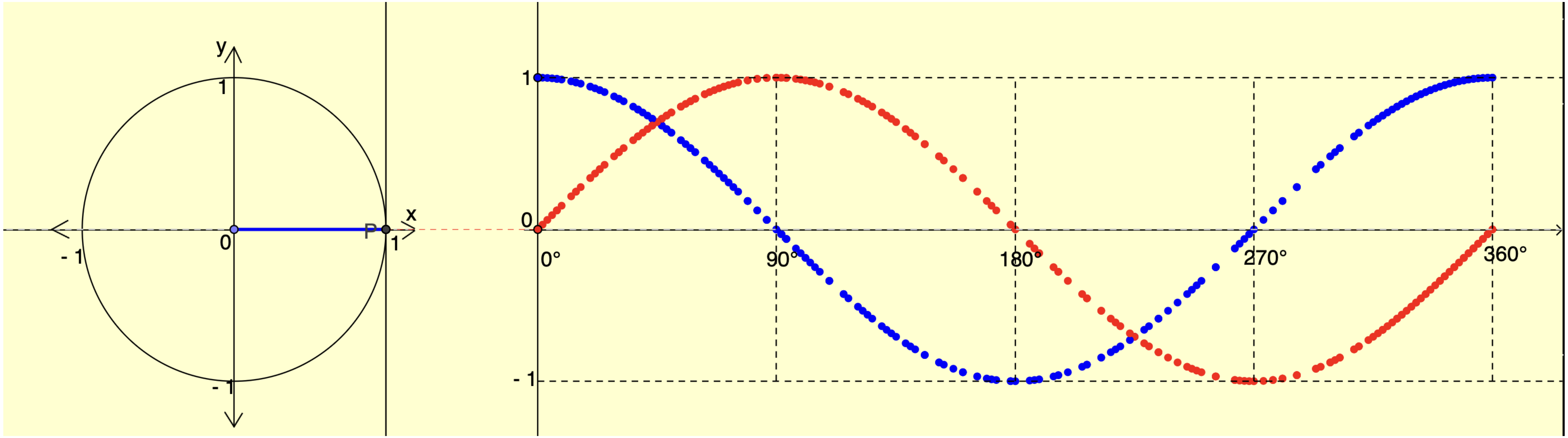
- Functions for which the input is an angle (the measure of rotation).
- The output of the function is a ratio… why?

Graphing Trigonometric Functions

Graphing Trigonometric Functions

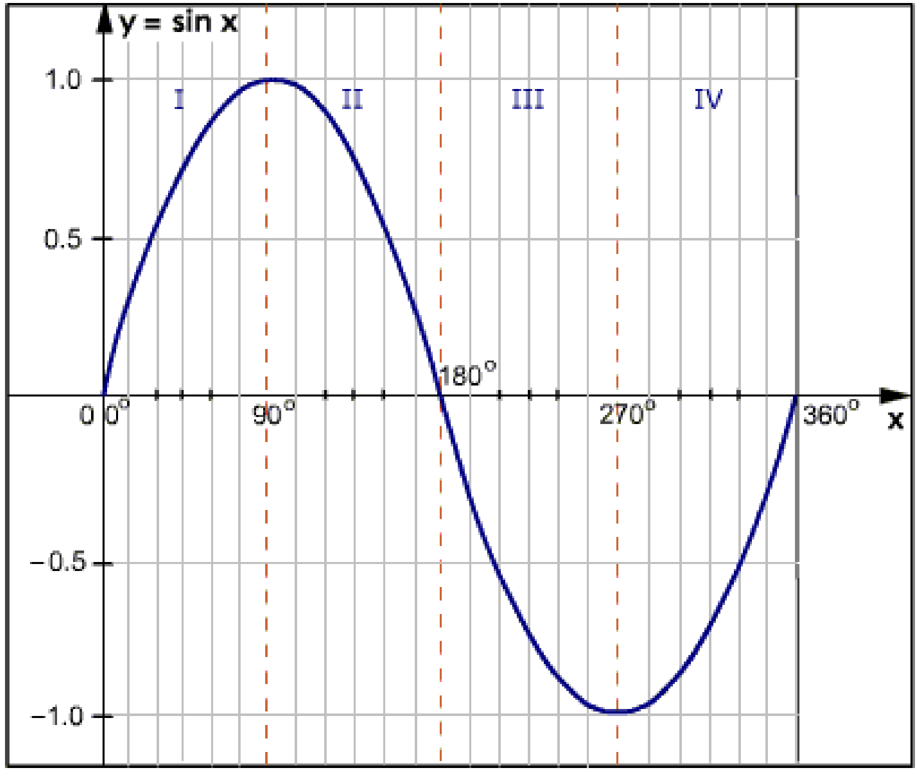
https://www.geogebra.org/m/UjjwuM8p

The values of $\cos(\alpha)$ and $\sin(\alpha)$ in the four quadrants



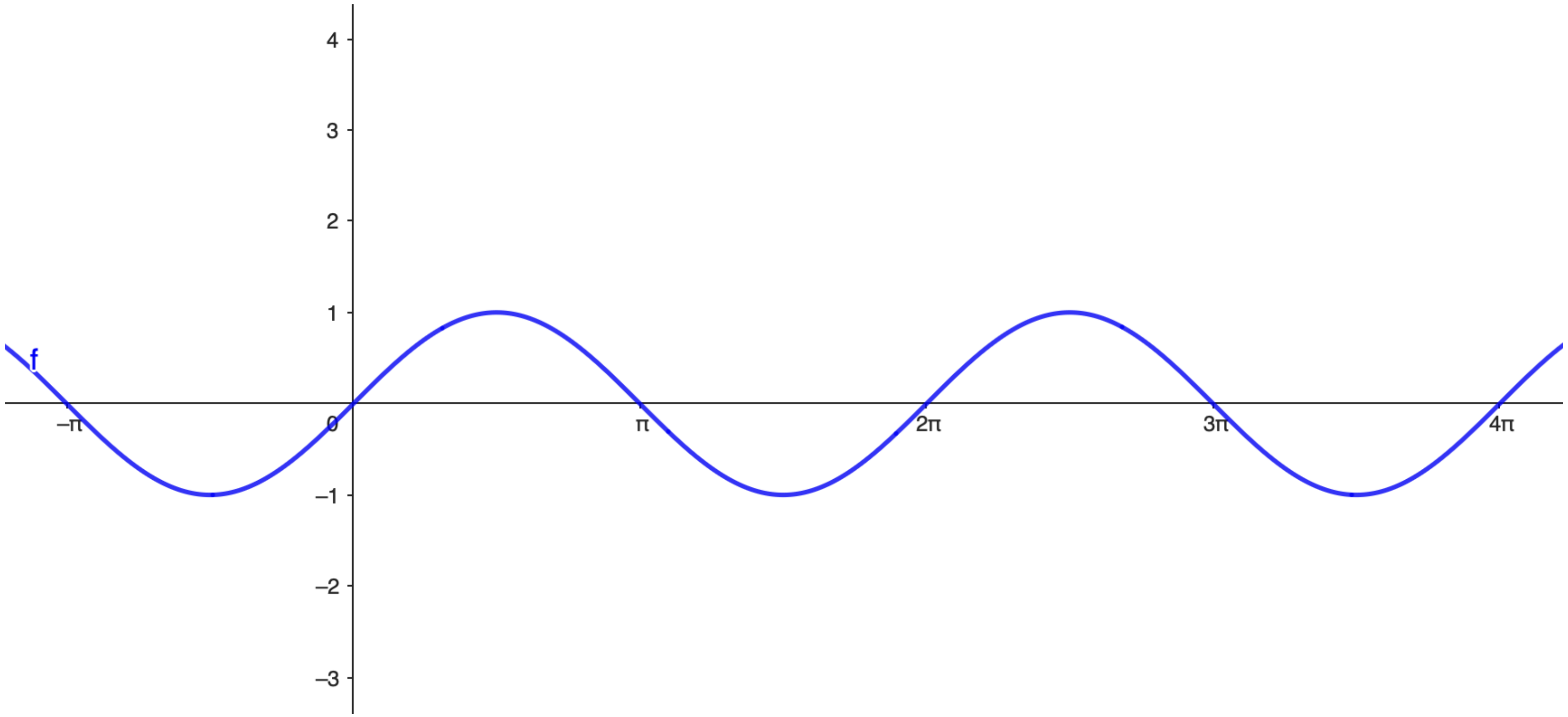
Graph of $y = \sin\ x$

x	0	30	60	90	120	150	180	210	240	270	300	330	360
$y = \sin x$	0	0.5	0.8	1	0.8	0.5	0	-0.5	-0.8	-1	-0.8	-0.5	0



Graph of $y = \sin\ x$: Additional Info

$f(\alpha) = \sin(x)$



Graph of $y = \sin\ x$: Additional Info

If values outside 0 to 360 degrees are calculated, the graph will start to repeat itself.

Hence, we can say the Period (where the graph starts to repeat itself) is: $2\pi = 360°$

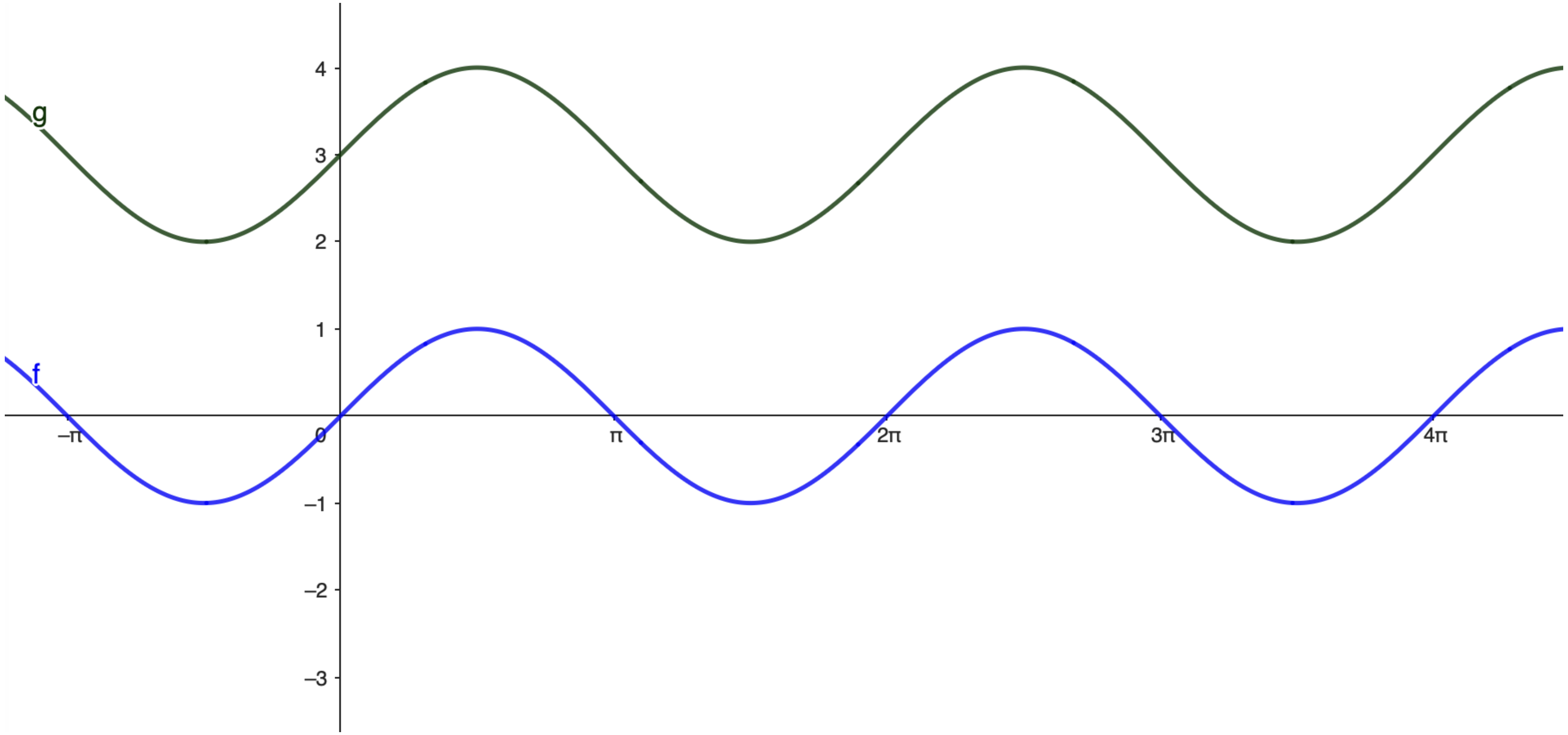
The maximum value of $\sin x$ is 1

The minimum value of $\sin x$ is -1

The Range is [-1,1]

Potential Student Task

Consider the function $g(\alpha)$ below:

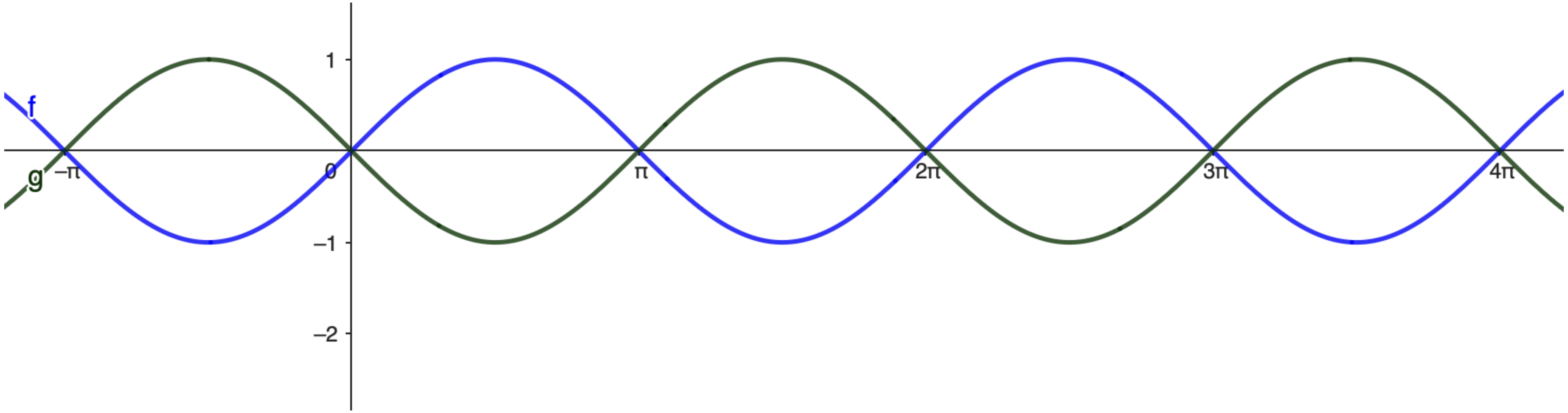


Which of the following represents $g(\alpha)$:

(a) $\sin(3\alpha)$ (b) $3 + \sin(\alpha)$ (c) $3\sin(\alpha)$

Potential Student Task

Consider the function $g(\alpha)$ below:



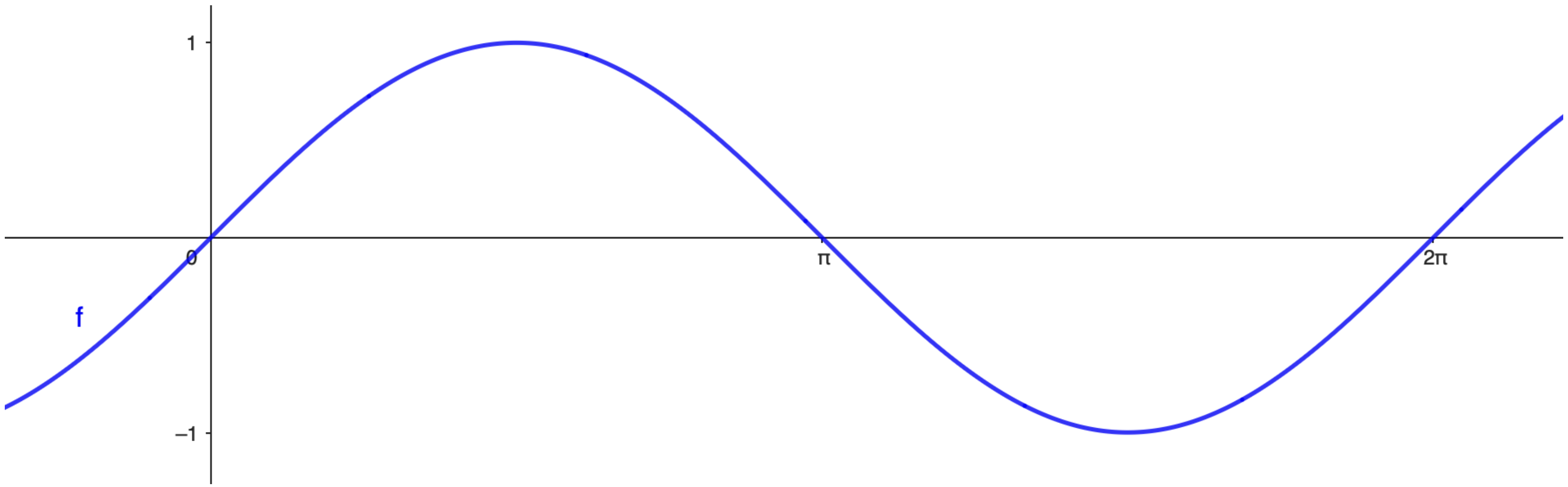
How does $g(\alpha)$ compare with $f(\alpha)$?

$$g\left(\frac{\pi}{2}\right) = -f\left(\frac{\pi}{2}\right)$$

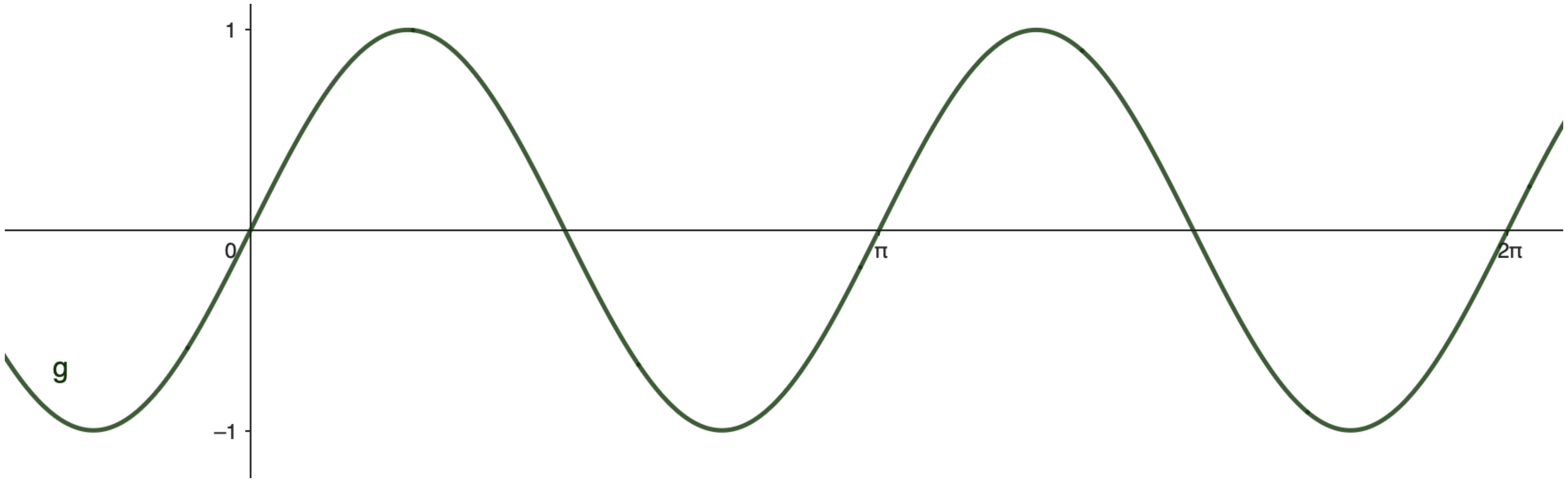
$$g(\alpha) = -f(\alpha)$$

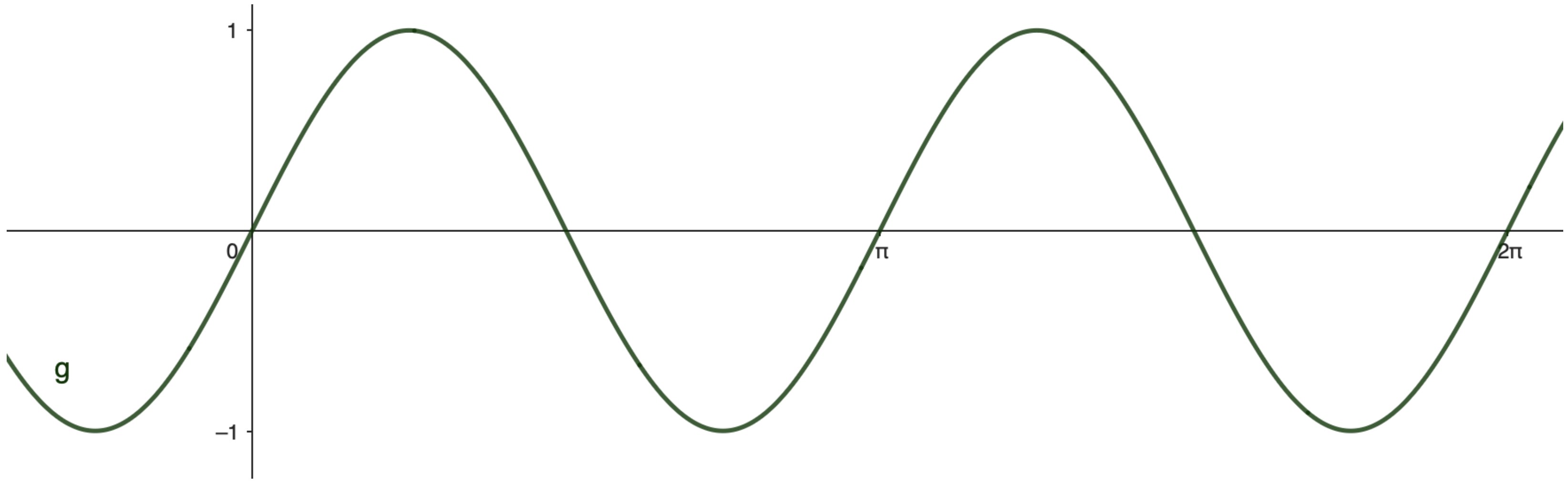
$$g(\alpha) = -\sin(\alpha)$$

Compare the functions $f(\alpha)$ and $g(\alpha)$ below:



What is the period of $g(\alpha)$:





The period of the function $\sin(\alpha) = 2\pi$

The period of the function $g(\alpha) = \pi$

$g(\alpha) = \sin(A(\alpha))$

In the function, $\sin(A\,(\alpha))$ when $\alpha = \pi$, the function repeats – this marks the period of the function.

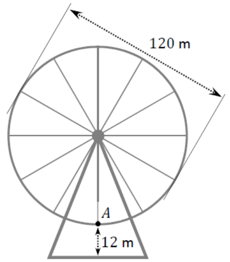
Therefore, $\sin(A\,(\pi)) = \sin(2\,\pi)$

$A = 2$

$g(\alpha) = \sin(2\alpha)$

Examine the questions; try them.

Question 8 **(50 marks)**



A Ferris wheel has a diameter of 120 m.
When it is turning, it completes exactly 10 full rotations in one hour.
The diagram above shows the Ferris wheel before it starts to turn.
At this stage, the point A is the lowest point on the circumference of the wheel, and it is at a height of 12 m above ground level.

The height, h, of the point A after the wheel has been turning for t minutes is given by:

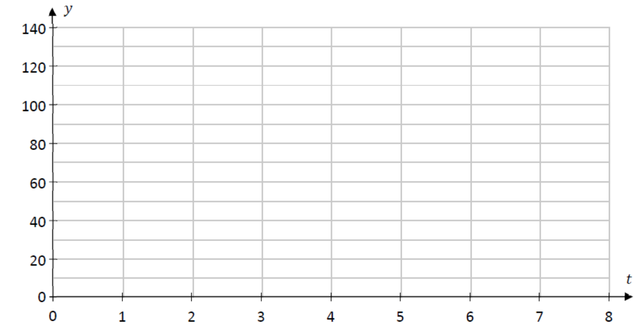
$$h(t) = 72 - 60\cos\left(\frac{\pi}{3}t\right)$$

where h is in metres, $t \in \mathbb{R}$, and $\frac{\pi}{3}t$ is in radians.

(a) Complete the table below. The value of $h(1)$ is given.

t	0	1	2	3	4	5	6	7	8
$h(t)$		42							

(b) Draw the graph of $y = h(t)$ for $0 \le t \le 8$, $t \in \mathbb{R}$.



(c) Find the period and range of $h(t)$.

Period = ______________ Range = [,]

(d) During a 50-minute period, what is the **greatest** number of minutes for which the point A could be higher than 42 m?

This question continues on the next page.

(e) By solving the following equation, find the **second** time (value of t) that the point A is at a height of 110 m, after it starts turning:

$$72 - 60\cos\left(\frac{\pi}{3}t\right) = 110$$

Give your answer in minutes, correct to 2 decimal places.

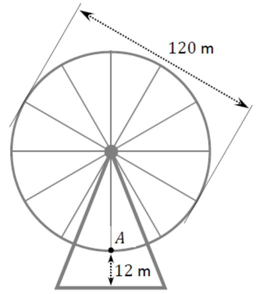
(f) Use integration to find the average height of the point A over the first 8 minutes that the wheel is turning. Give your answer correct to 1 decimal place.

Remember that $h(t) = 72 - 60\cos\left(\frac{\pi}{3}t\right)$.

If you are unsure, make a list of questions, identifying what you would need to know to try the questions.

Examine the worked solutions; consider each approach and provide a rationale for why each step was carried out

Question 8 **(50 marks)**



A Ferris wheel has a diameter of 120 m.
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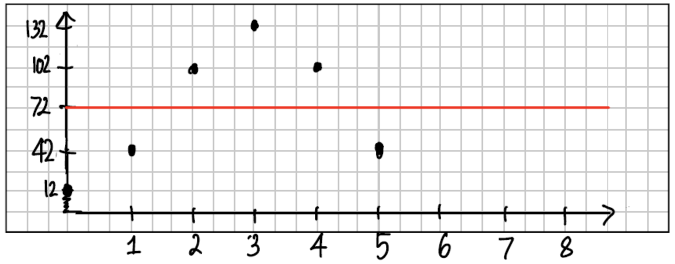
$$h(t) = 72 - 60\cos\left(\frac{\pi}{3}t\right)$$

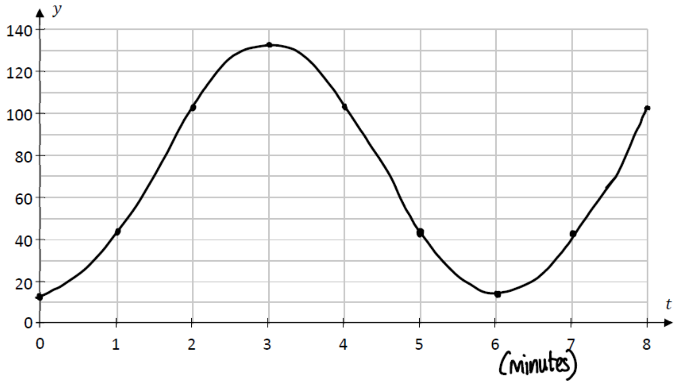
where h is in metres, $t \in \mathbb{R}$, and $\frac{\pi}{3}t$ is in radians.

(a) Complete the table below. The value of $h(1)$ is given.

t	0	1	2	3	4	5	6	7	8
$h(t)$	12	42	102	132	102	42	12	42	102



(b) Draw the graph of $y = h(t)$ for $0 \le t \le 8$, $t \in \mathbb{R}$.



(c) Find the period and range of $h(t)$.

$\frac{2\pi}{\left(\frac{\pi}{3}\right)} = \frac{2\pi}{1} \times \frac{3}{\pi}$

$= 6$

Period = 6

Range = [12 , 132]

(d) During a 50-minute period, what is the **greatest** number of minutes for which the point A could be higher than 42 m?

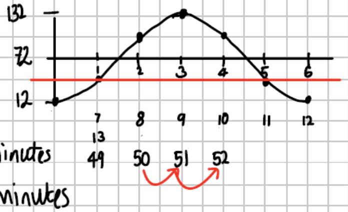
$\frac{50}{6} = 8\frac{1}{3}$

4 mins in 1 period

$8(4) = 32$ minutes

$+ \; 2$ minutes

34



This question continues on the next page.

(e) By solving the following equation, find the **second** time (value of t) that the point A is at a height of 110 m, after it starts turning:

$$72 - 60\cos\left(\frac{\pi}{3}t\right) = 110$$

Give your answer in minutes, correct to 2 decimal places.

$72 - 60\cos\left(\frac{\pi}{3}t\right) = 110$

$-60\cos\left(\frac{\pi}{3}t\right) = 110 - 72$

$-60\cos\left(\frac{\pi}{3}t\right) = 38$

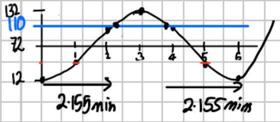
$\cos\left(\frac{\pi}{3}t\right) = \frac{38}{-60}$

$\frac{\pi}{3}(t) = \cos^{-1}\left(\frac{38}{-60}\right)$

$t_1 = \frac{\cos^{-1}\left(\frac{38}{-60}\right)}{\left(\frac{\pi}{3}\right)}$

$t_1 = \frac{3\left[\cos^{-1}\left(-\frac{38}{60}\right)\right]}{\pi}$

$t_1 = 2.154941337$



$\therefore t_2 = 6 - 2.155$

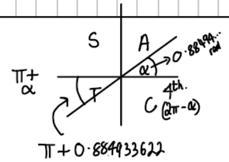
$= 3.845$

$= 3.85$ mins

$\cos(\alpha) = \frac{38}{60}$

$\alpha = \cos^{-1}\left(\frac{38}{60}\right)$

$\alpha = 0.884933622$



$\cos\left(\frac{\pi}{3}t\right) = \frac{38}{-60}$

$\cos(\pi + 0.884933622) = -\frac{38}{60}$

$\frac{\pi}{3}t_2 = \pi + 0.884933622$

$t_2 = \frac{3(\pi + 0.884933622)}{\pi} = 3.845058663$

$t_2 = 3.85$ mins

Examine the worked solutions; consider each approach and provide a rationale for why each step was carried out

(f) Use integration to find the average height of the point A over the first 8 minutes that the wheel is turning. Give your answer correct to 1 decimal place.

Remember that $h(t) = 72 - 60\cos\left(\frac{\pi}{3}t\right)$.

$$\frac{1}{8-0}\int_0^8 72 - 60\cos\left(\frac{\pi}{3}t\right)\,dt$$

$$\frac{1}{8}\left[72t - \frac{60\sin\left(\frac{\pi}{3}\right)t}{\frac{\pi}{3}}\right]_{t=0}^{t=8}$$

$$\frac{1}{8}\left[72(t) - \frac{3\left[60\sin\left(\frac{\pi}{3}t\right)\right]}{\pi}\right]_{t=0}^{t=8}$$

$$\left[\frac{72t}{8} - \frac{180\sin\left(\frac{\pi}{3}t\right)}{8\pi}\right]_{t=0}^{t=8}$$

$$\left[9t - \frac{45\sin\left(\frac{\pi}{3}t\right)}{2\pi}\right]$$

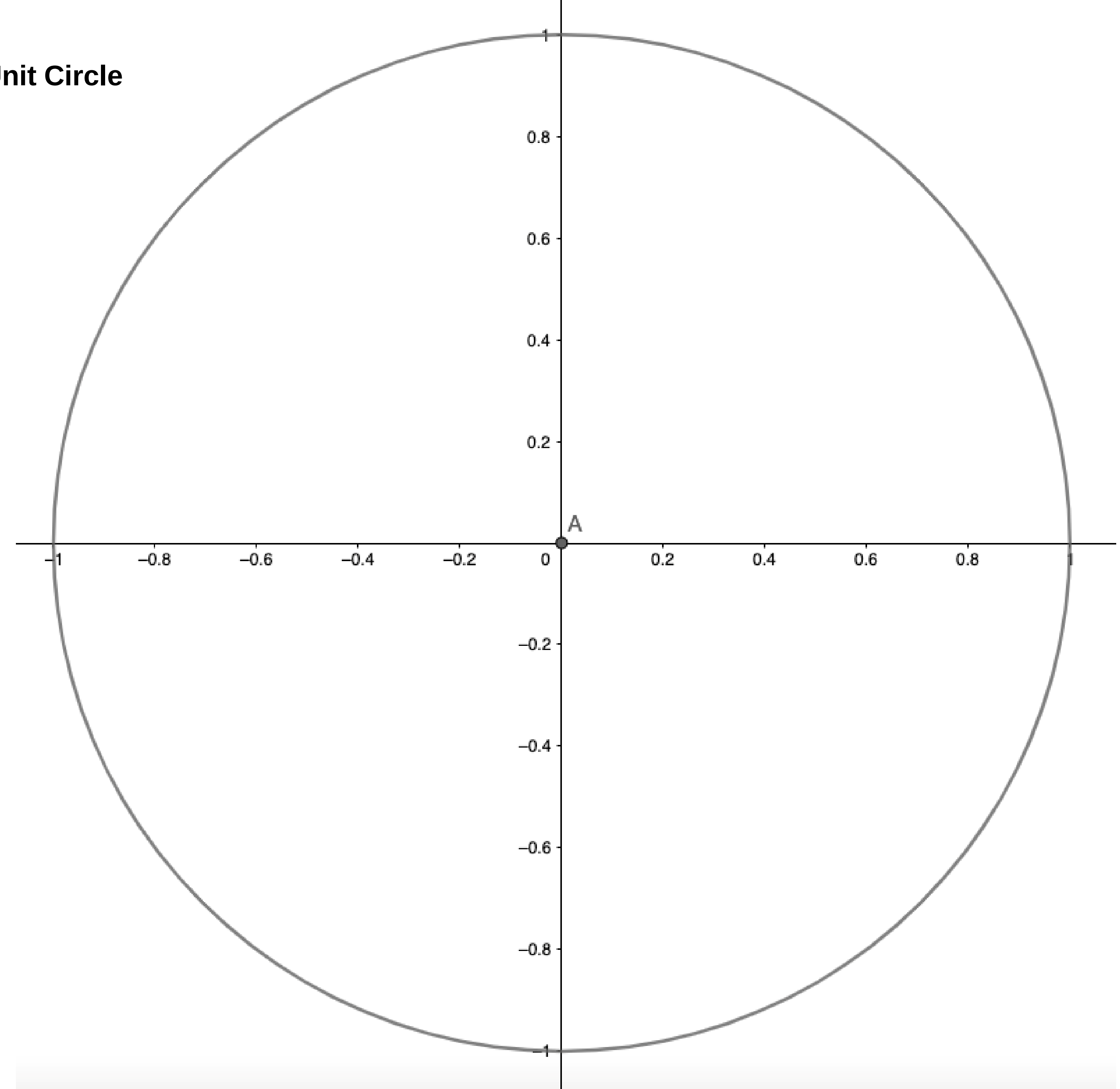
$$\left[9(8) - \frac{45\sin\left(\frac{8\pi}{3}\right)}{2\pi}\right] - \left[9(0) - \frac{45\sin\left(\frac{\pi}{3}(0)\right)}{2\pi}\right]$$

$$= 72 - \frac{45\sin\left(\frac{8\pi}{3}\right)}{2\pi}$$

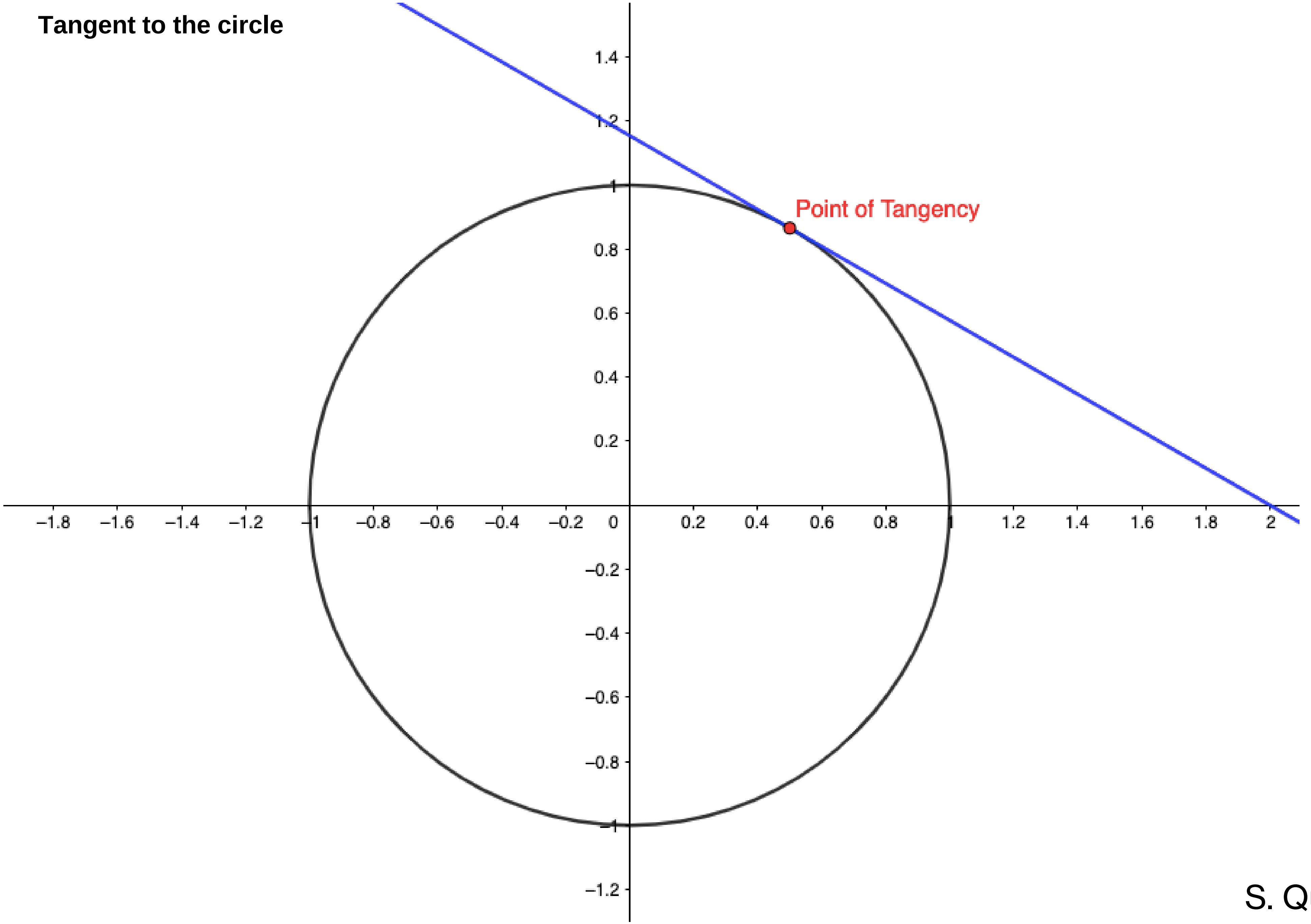
$$= 68.79754993$$

$$\approx 68.8 \text{ metres.}$$

The Unit Circle

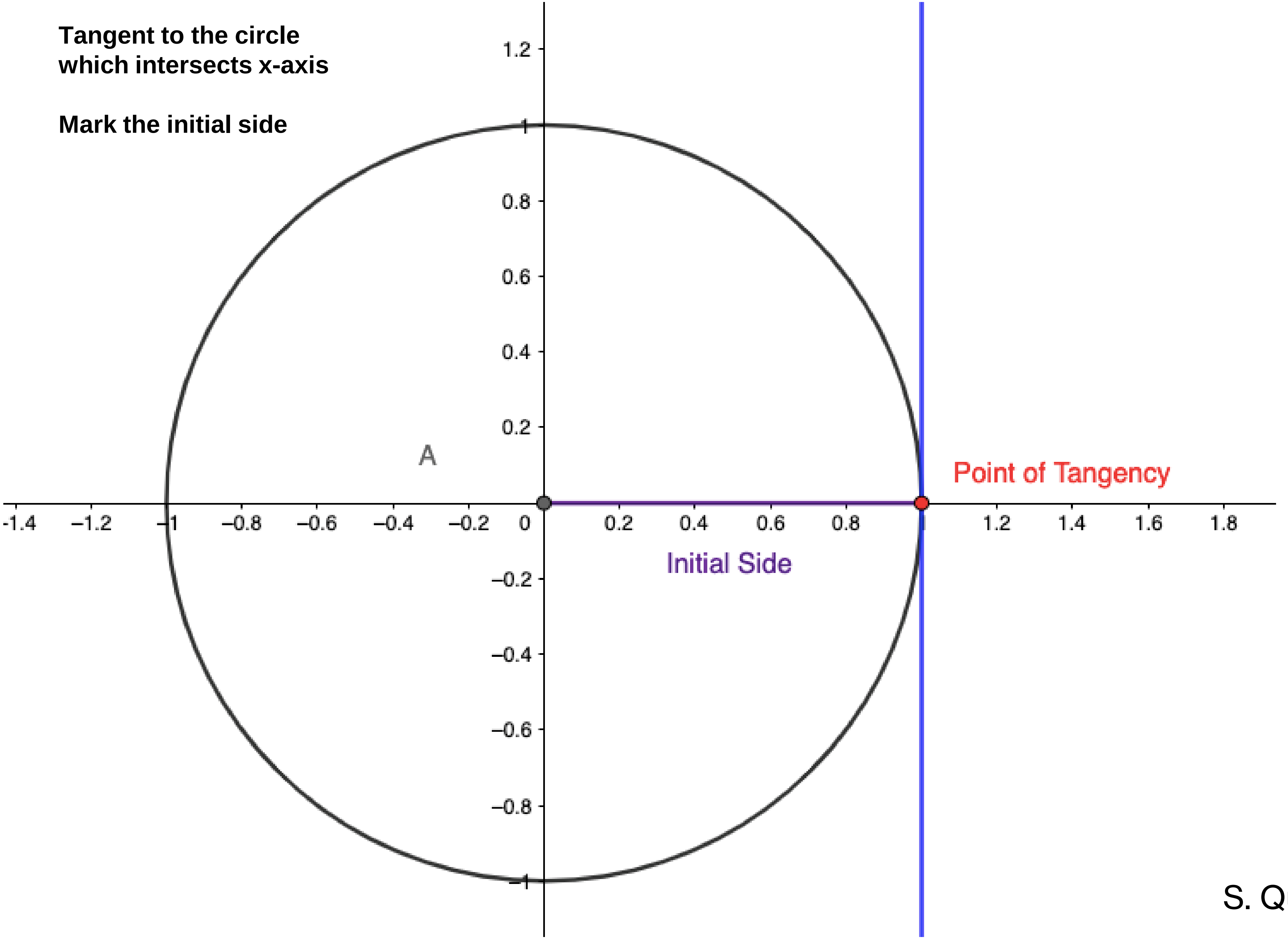


Tangent to the circle

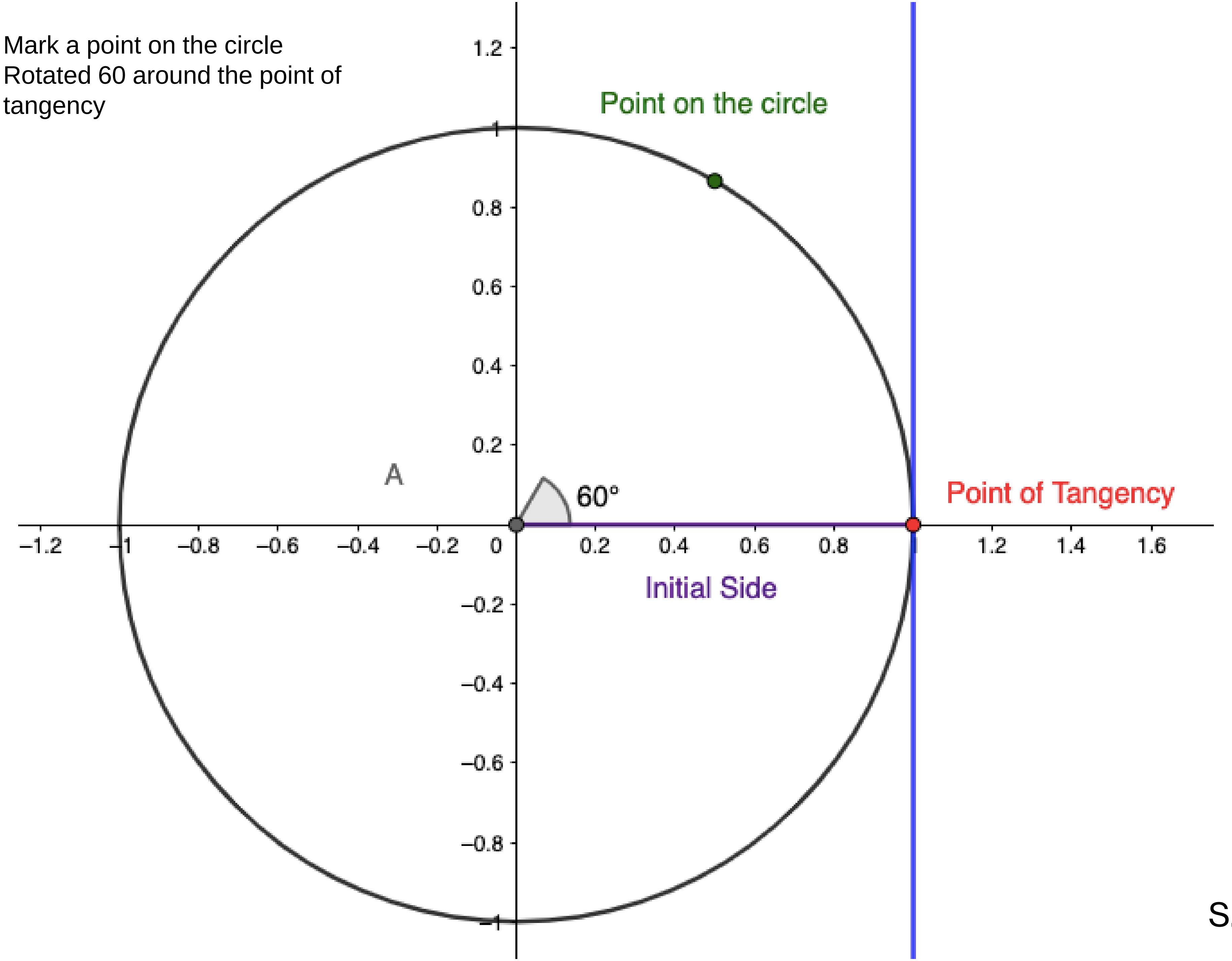


Tangent to the circle which intersects x-axis

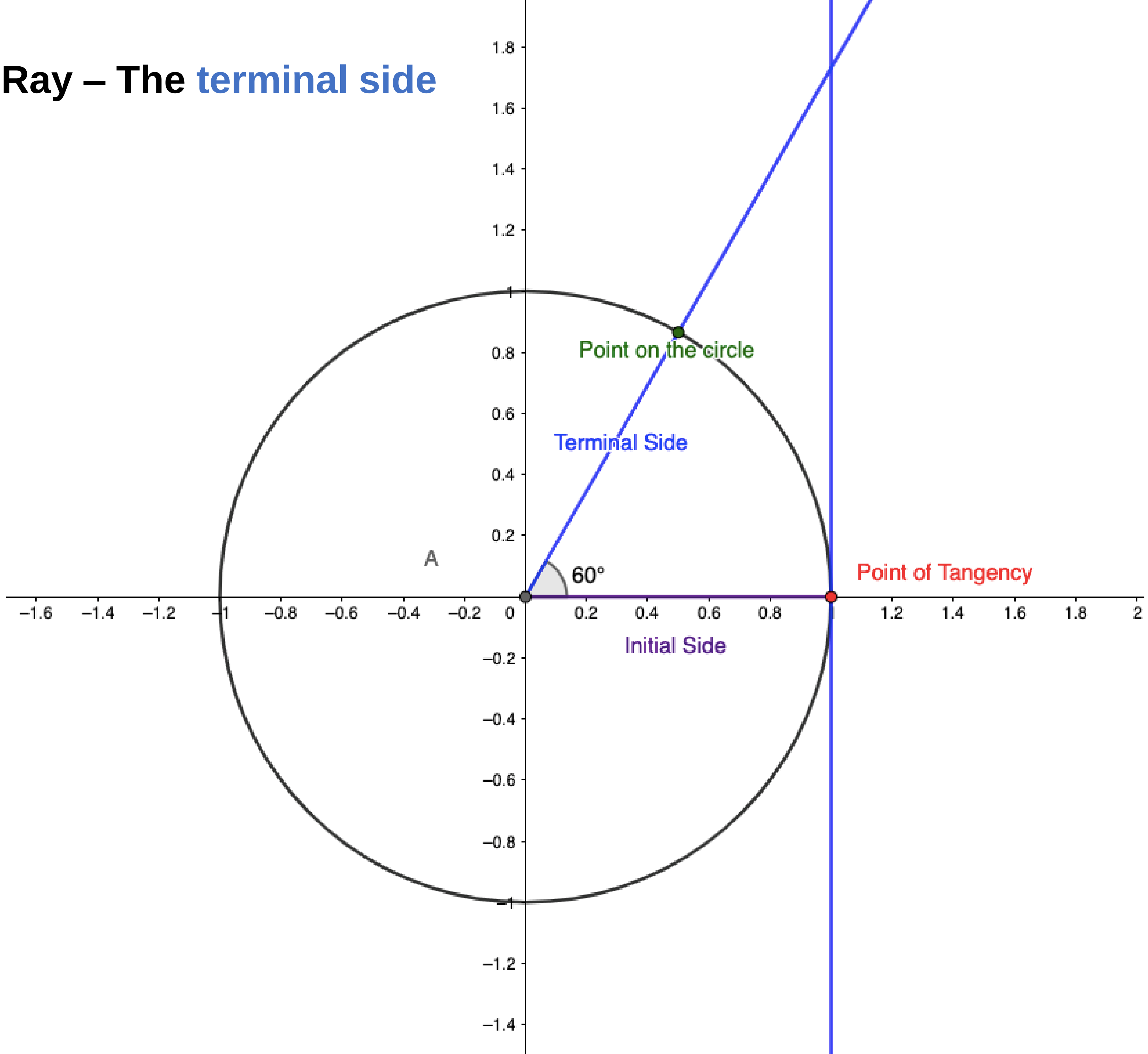
Mark the initial side



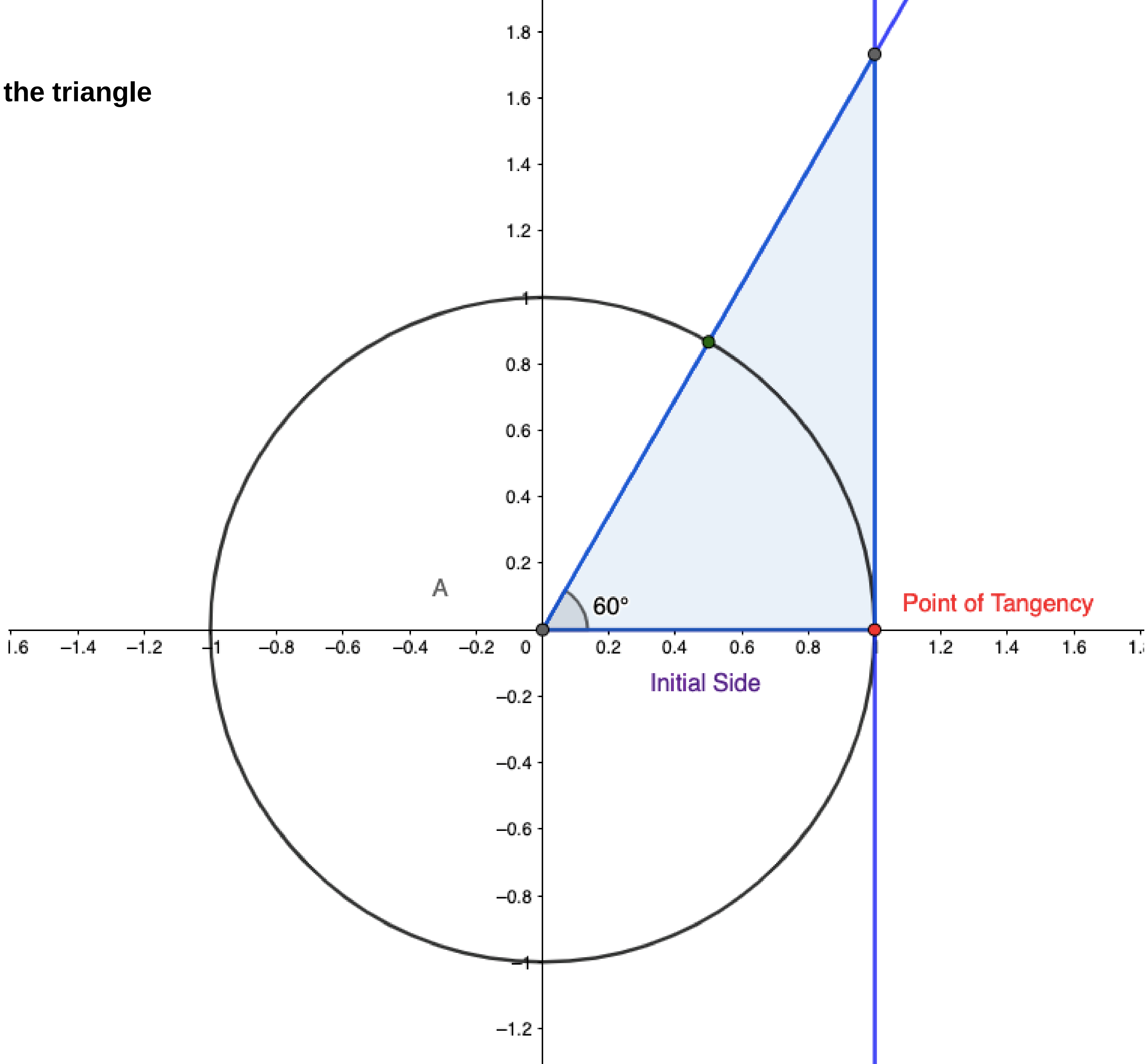
Mark a point on the circle
Rotated 60 around the point of
tangency

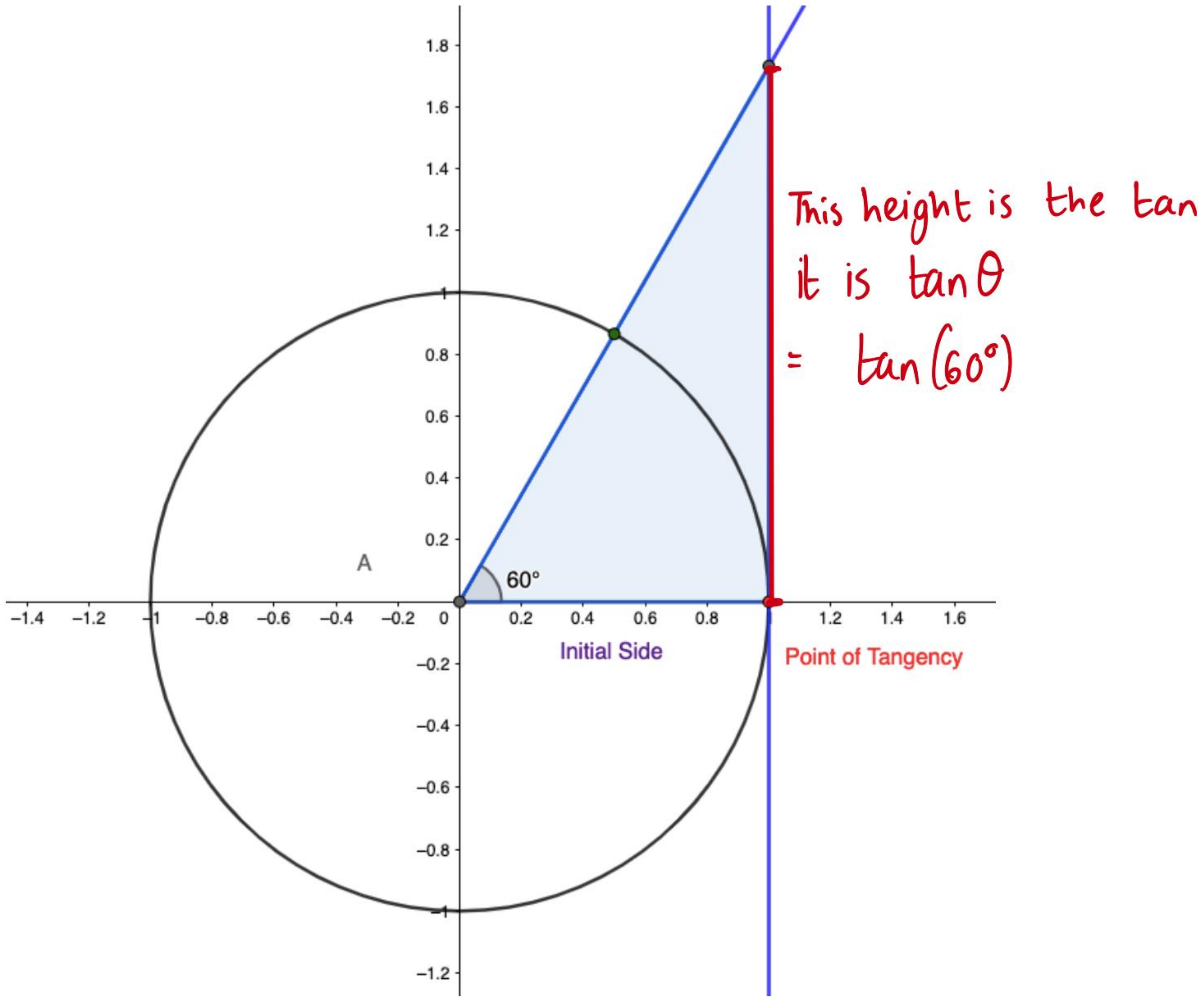


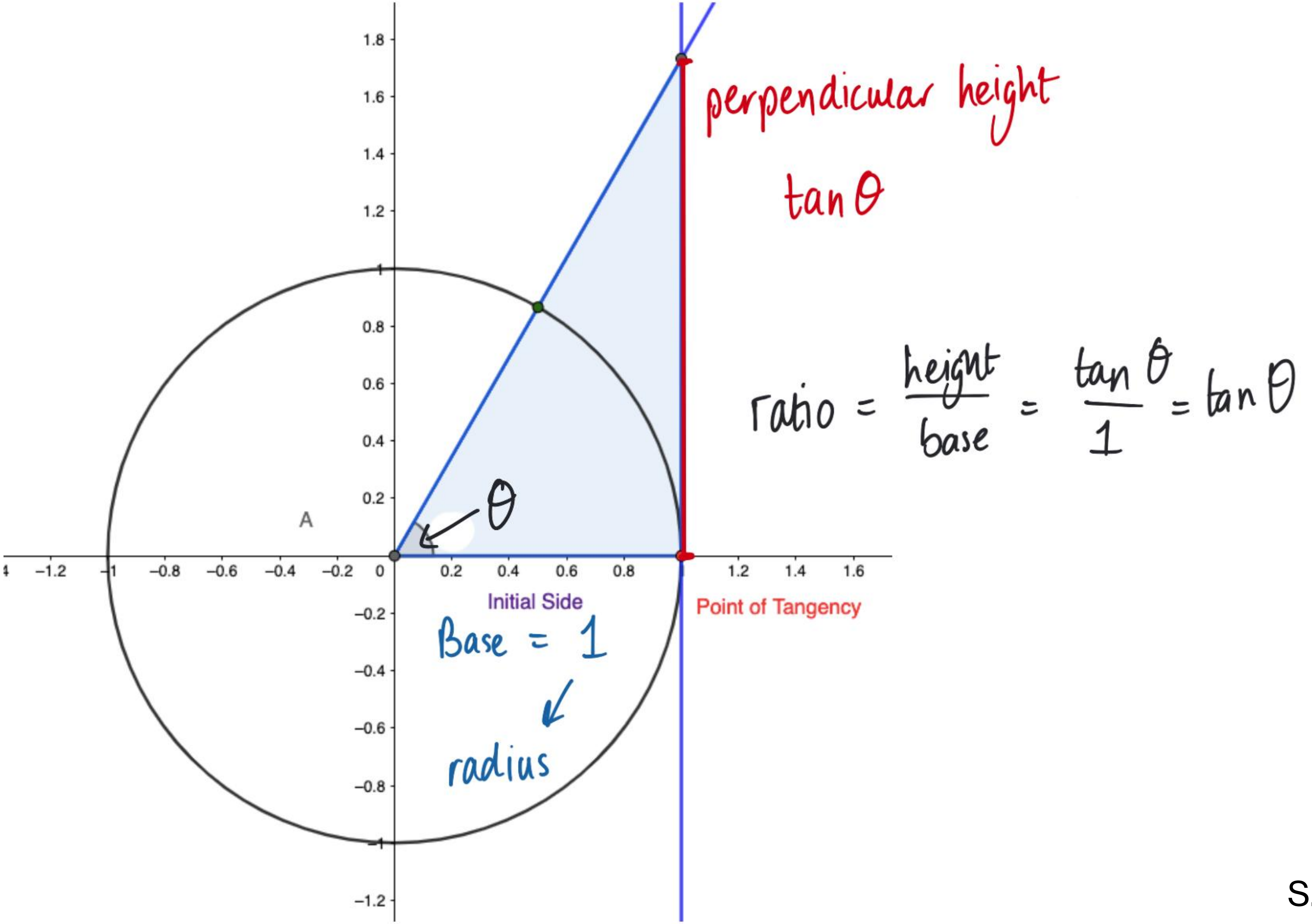
Draw Ray – The terminal side



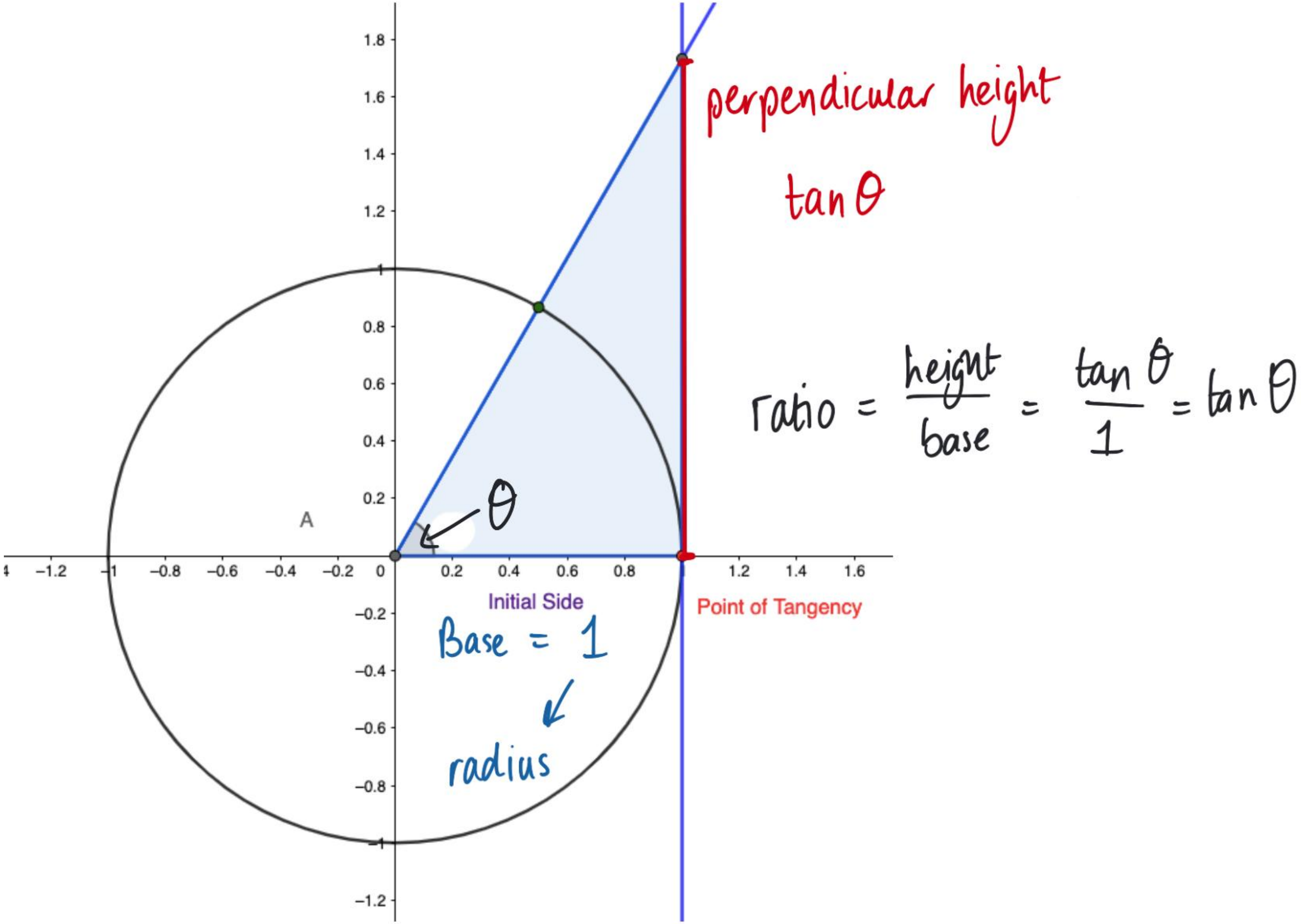
Draw the triangle




1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
A
60°
−1.4
−1.2
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1.2
1.4
1.6
Initial Side
Point of Tangency
−0.2
−0.4
−0.6
−0.8
−1
−1.2
This height is the tan
it is tan θ
= tan(60°)


perpendicular height
tan θ
ratio = height / base = tan θ / 1 = tan θ
A
θ
Initial Side
Point of Tangency
Base = 1
radius

We could draw a similar right angle triangle within the unit circle



Similar triangle within the unit circle

What is the height of the opposite side in the smaller similar triangle?

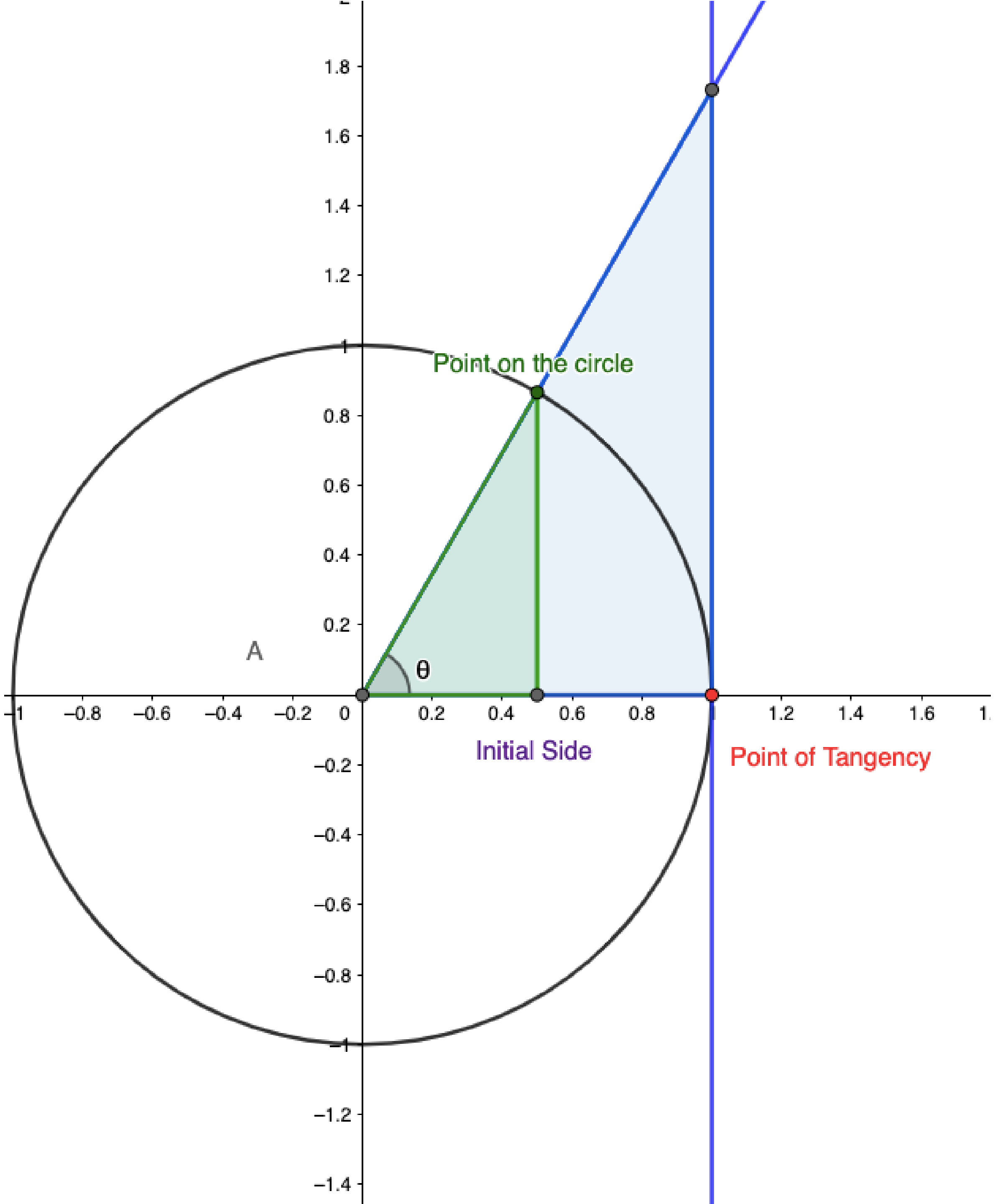
$$\sin\theta$$

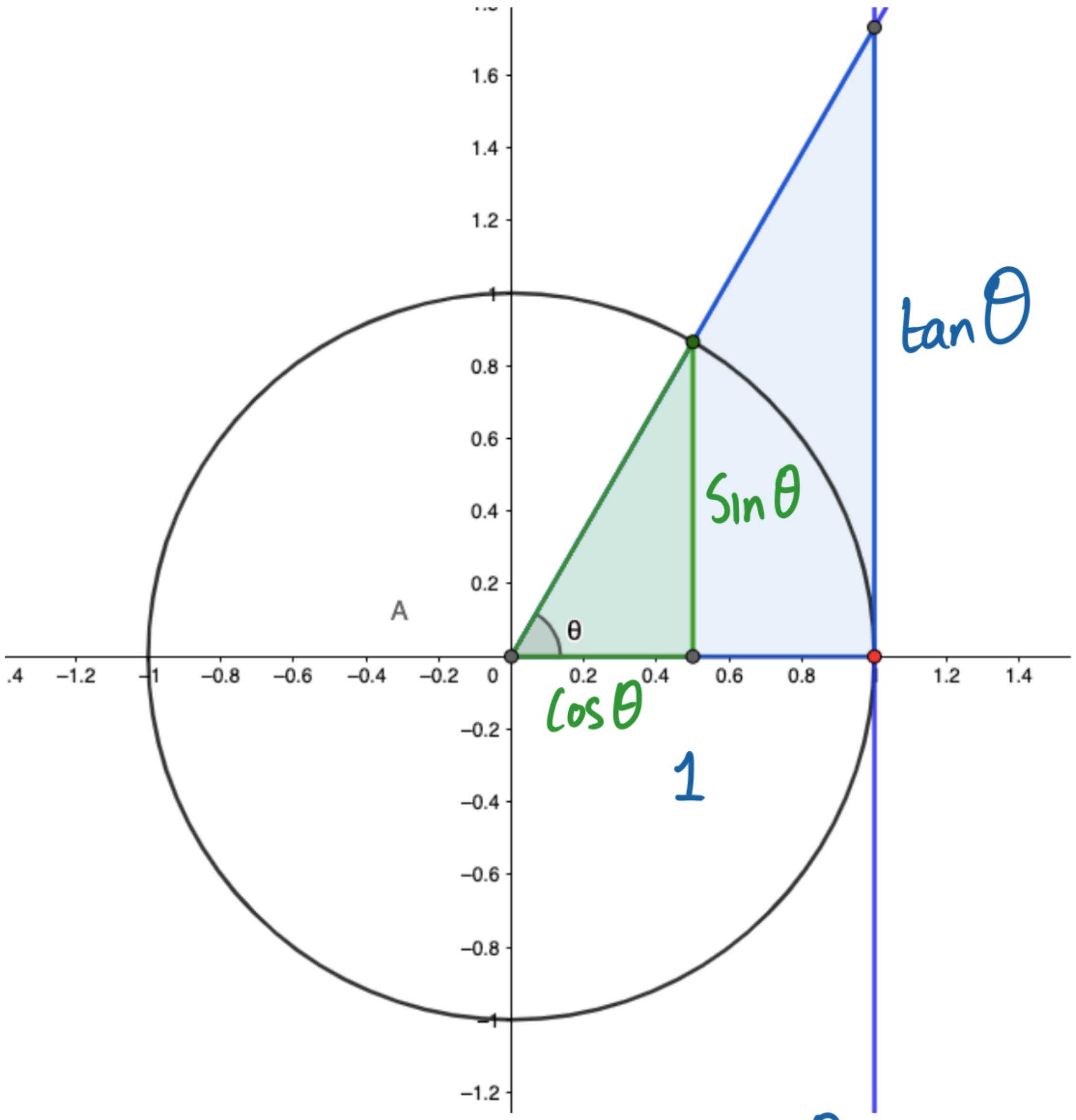
What is the length of the base side in the smaller similar triangle?

$$\cos\theta$$

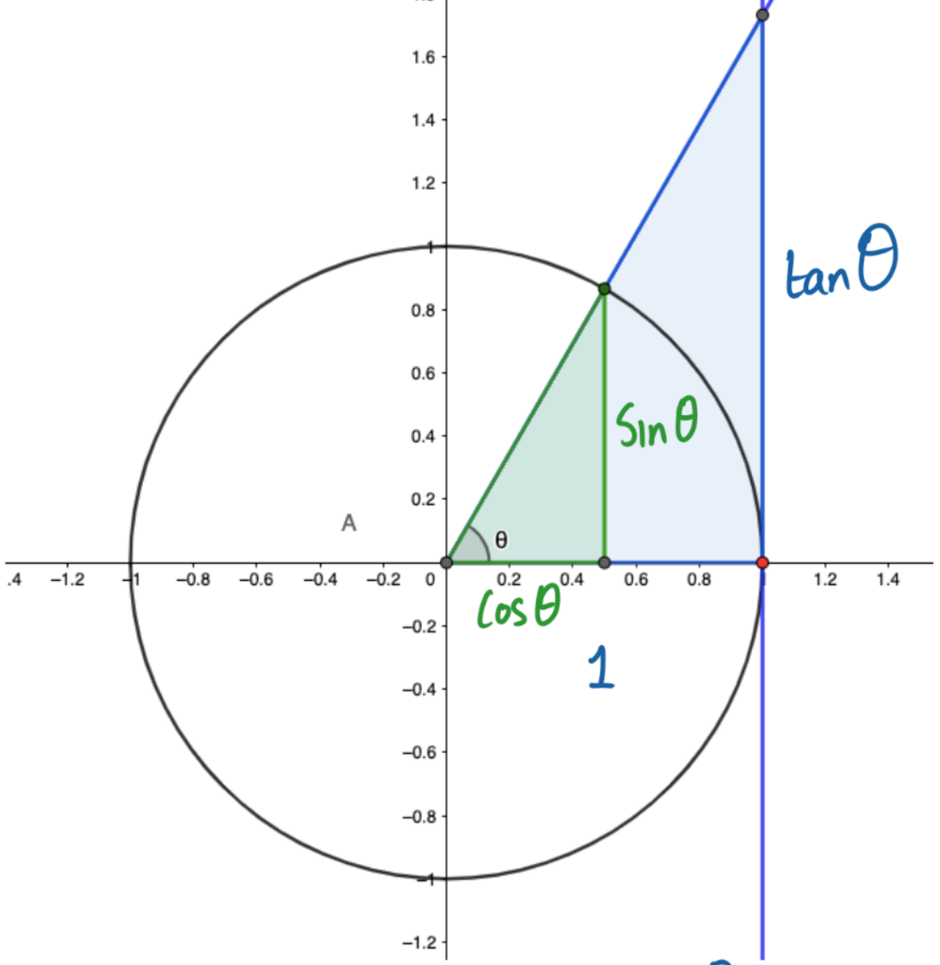
In the smaller triangle what is the ratio of the height to the base

$$\frac{\sin\theta}{\cos\theta}$$



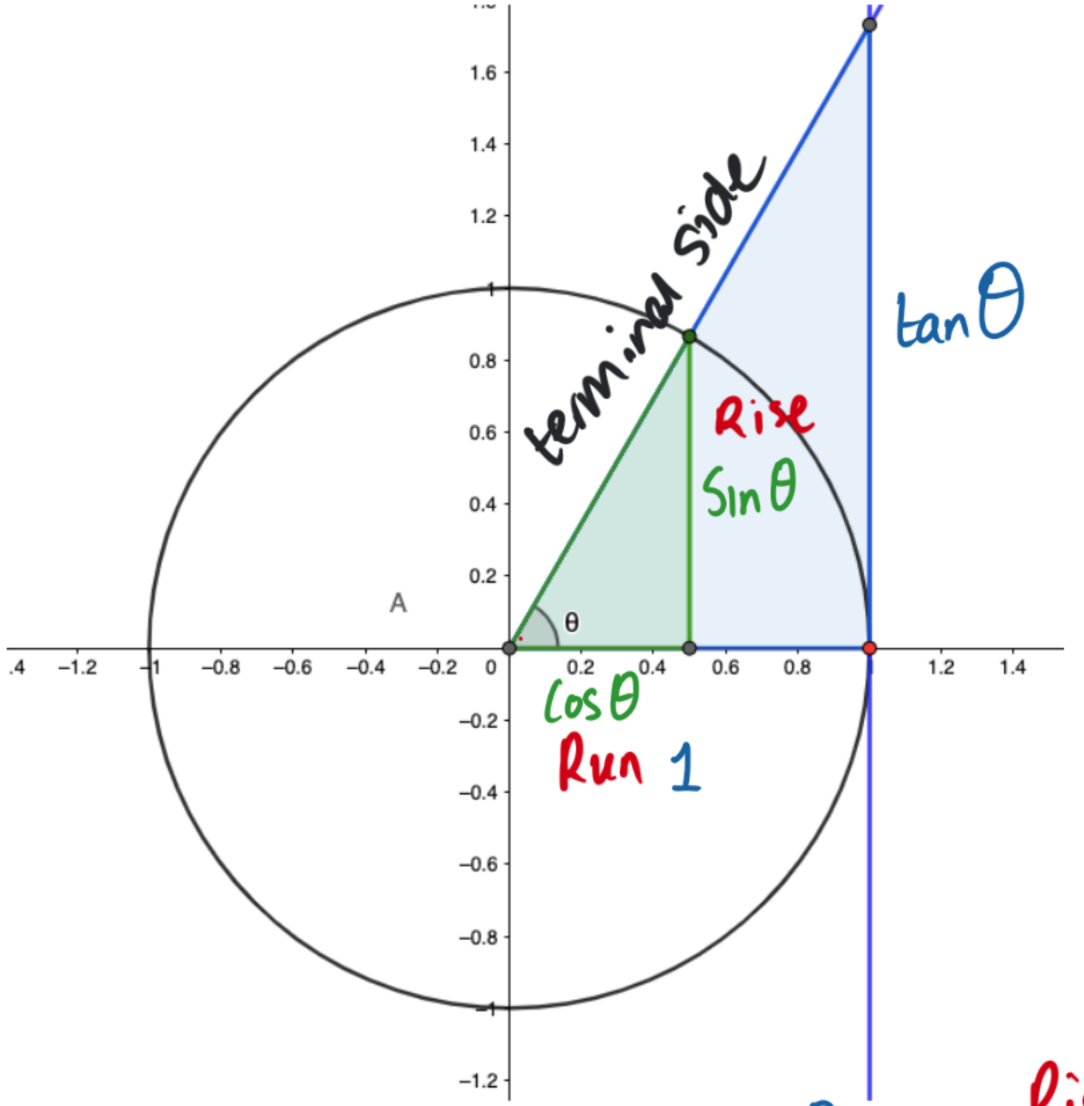


ratio : $\dfrac{\text{height}}{\text{base}} = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\tan\theta}{1}$

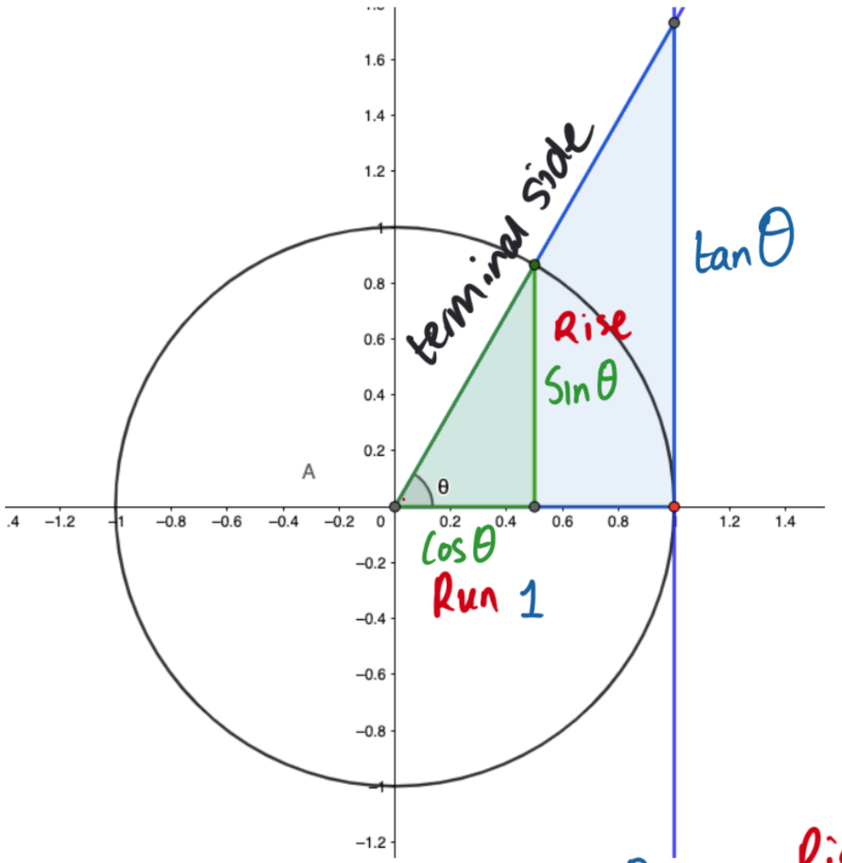


$$\text{ratio}: \frac{\text{height}}{\text{base}} = \frac{\sin\theta}{\cos\theta} = \frac{\tan\theta}{1}$$

$$\frac{\sin\theta}{\cos\theta} = \tan\theta$$



$$\text{ratio}: \frac{\text{height}}{\text{base}} = \frac{\sin\theta}{\cos\theta} = \frac{\tan\theta}{1} = \frac{\text{Rise}}{\text{Run}} = \text{Slope of terminal side}$$



opposite

$$\text{ratio}: \frac{\text{height}}{\text{base}} = \frac{\sin\theta}{\cos\theta} = \frac{\tan\theta}{1} = \frac{\text{Rise}}{\text{Run}}$$

adjacent

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \text{Slope of terminal side}$$

The key ideas of fractions:
- A fraction as a ratio of two numbers

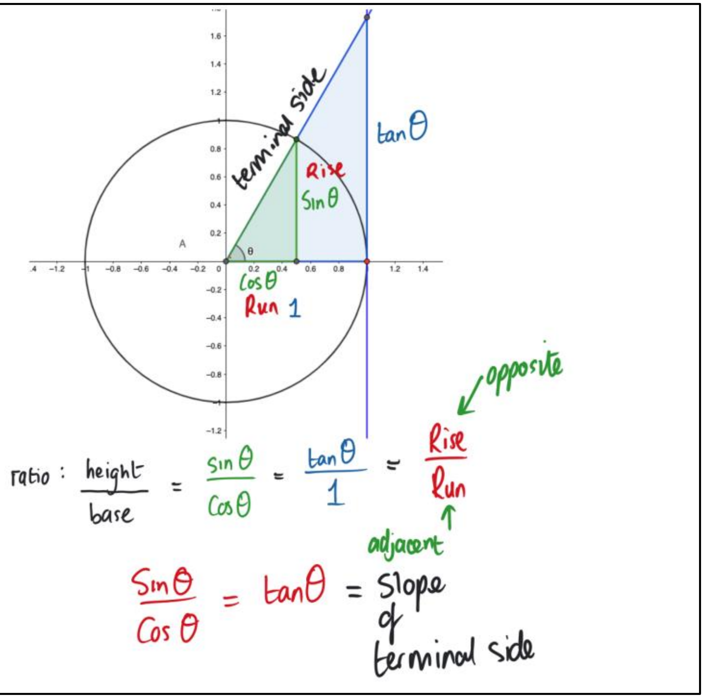
Task:

Explore what does 6 ÷ 2 mean..

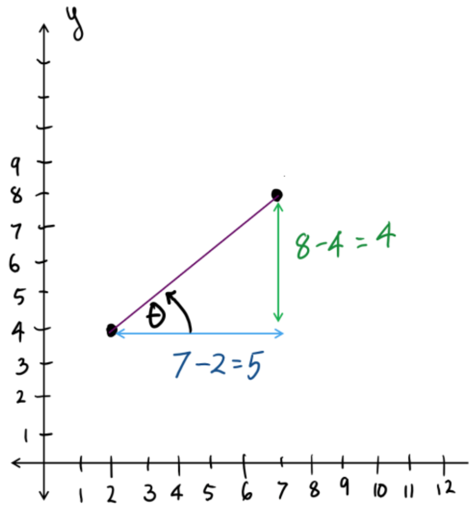
Find two other numbers which have the same relationship

What about $\frac{1}{2}$ and the number 20? How do they compare?

Find two other numbers that have the same relationship



Create and **seize** opportunities of learning

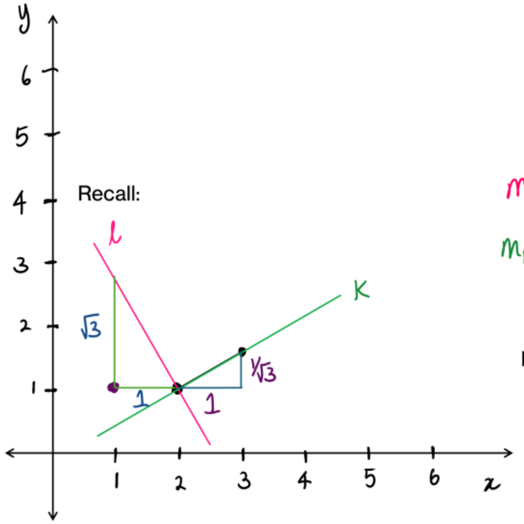


Find the angle of this gradient?

$$\tan(\theta) = \frac{4}{5}$$

$$\theta = \tan^{-1}\left(\frac{4}{5}\right)$$

$$\theta \approx 38.67^\circ$$



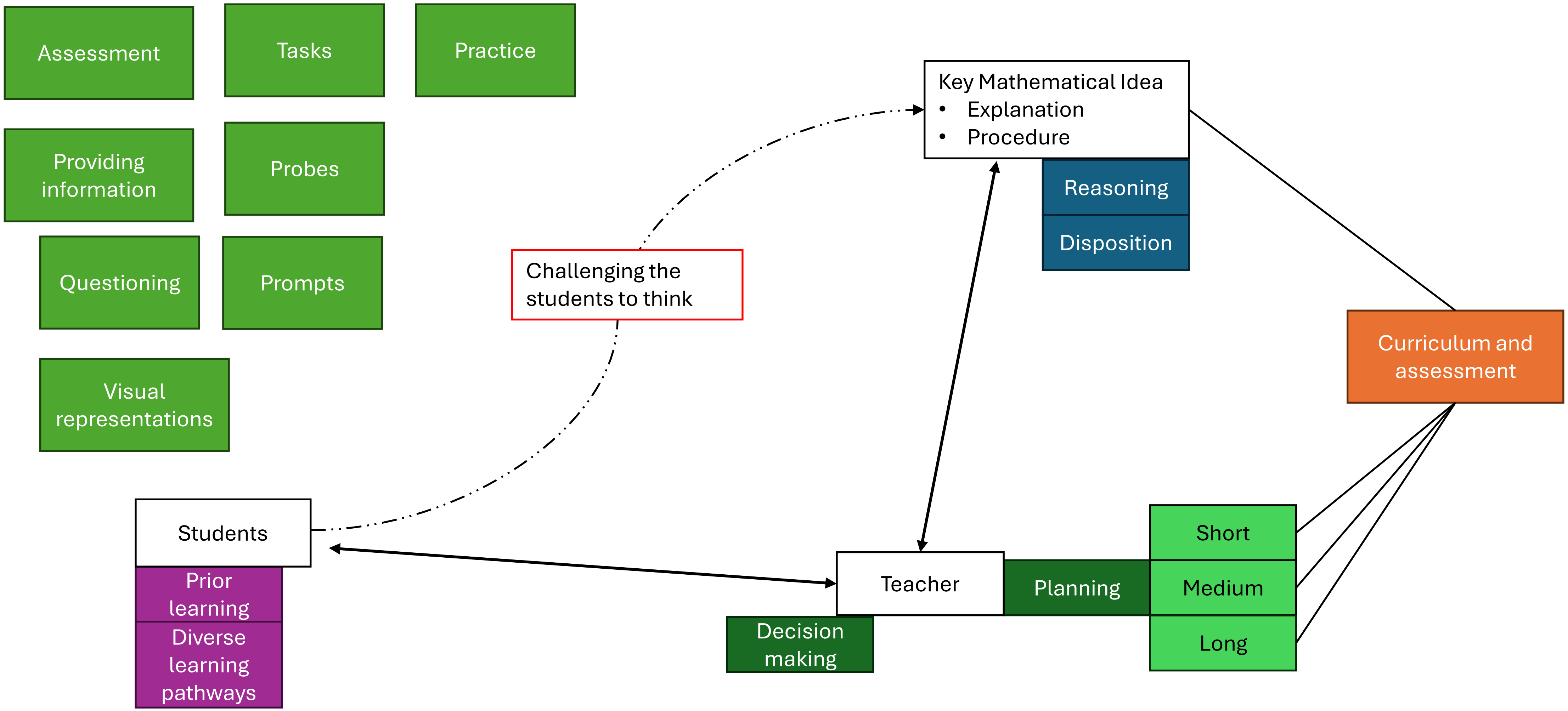
Find the slopes of the lines, l and k

$$m_l = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$

$$m_k = \frac{\frac{1}{\sqrt{3}}}{1} = \frac{1}{\sqrt{3}}$$

Find the product of the slopes of the lines, l and k

$$\frac{1}{\sqrt{3}} \cdot -\sqrt{3} = \frac{-\sqrt{3}}{\sqrt{3}} = -1$$


Assessment
Tasks
Practice
Providing information
Probes
Questioning
Prompts
Visual representations
Challenging the students to think
Key Mathematical Idea
• Explanation
• Procedure
Reasoning
Disposition
Curriculum and assessment
Students
Prior learning
Diverse learning pathways
Teacher
Planning
Decision making
Short
Medium
Long

To conclude…

- Teaching for ***relational*** understanding versus ***instrumental*** understanding…
- Promoting ***content-level*** understanding or ***concept-level*** understanding.
- Creating opportunities for ***problem-solving*** *level* understanding
- Seizing opportunities for enabling learners to **come to ‘see**’ *why and how.*

Thank you!!

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